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Rigidity for von Neumann algebras given by locally compact groups

Tobe Deprez

Supervisor:
Prof. dr. Stefaan Vaes

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Tobe DEPREZ
Research Assistent of the Research Foundation - Flanders (FWO)

Examination committee:
Prof. dr. Marco Zambon, chair
Prof. dr. Stefaan Vaes, supervisor
Prof. dr. Johan Quaegebeur
Prof. dr. Joeri Van der Veken
Prof. dr. Gábor Szabó
Prof. dr. Tim de Laat
(University of Münster)
Prof. dr. Kenny De Commer
(Vrije Universiteit Brussel)

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Uitgegeven in eigen beheer, Tobe Deprez, Celestijnenlaan 200B box 2400, B-3001 Leuven (Belgium)

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In my master thesis, four years ago, I mentioned the following quote from the preface of Wolfgang Woess' book *Random Walks on Infinite Graphs and Groups*. Since this quote still applies four years later, I will shamelessly use it again.

> *"Anyone who has written a book will have experienced the mysterious fact that a text of finite length may contain an infinity of misprints and mistakes, which apparently were not there during your careful proof-reading."*
>
> — W. Woess

I have done my best to correct every mistake or typo, but since I only corrected a finite number of mistakes, I can only conclude that there are still infinitely many left. Therefore, I humbly apologize to everyone who happens to find one.

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Abstract

Given a group G, the group von Neumann algebra $L(G)$ is defined as the w.o. closure of the linear span of the left regular representation $\{\lambda_g\}_{g \in G}$. The group measure space construction of Murray and von Neumann associates to every nonsingular action $G \curvearrowright (X, \mu)$ a von Neumann algebra $L^\infty(X) \rtimes G$. A central problem in the study of von Neumann algebras is the classification of $L(G)$ and $L^\infty(X) \rtimes G$ in terms of the underlying group G and the underlying action $G \curvearrowright (X, \mu)$, respectively. By defining Ozawa's class $\mathcal{S}$ for locally compact groups, we obtain the first rigidity and classification results for group von Neumann algebras and group measure space von Neumann algebras given by nondiscrete, locally compact groups.

Class $\mathcal{S}$ for discrete groups plays an important role in rigidity results for group von Neumann algebras and group measure space von Neumann algebras given by discrete groups. We define class $\mathcal{S}$ for locally compact groups and characterize locally compact groups in this class as groups having an amenable action on a boundary that is small at infinity, generalizing a theorem of Ozawa. We provide examples of locally compact groups in class $\mathcal{S}$ and prove that class $\mathcal{S}$ is closed under measure equivalence.

We prove that for arbitrary free, probability measure preserving actions of weakly amenable groups in class $\mathcal{S}$, the group measure space von Neumann algebra $L^\infty(X) \rtimes G$ has a unique Cartan subalgebra up to unitary conjugacy. We then deduce a W*-strong rigidity theorem of irreducible actions of products of such groups. These theorems in particular apply to connected, simple Lie groups of real rank one with finite center and groups acting properly on trees or hyperbolic graphs. We furthermore prove strong solidity results for the group von Neumann algebras of weakly amenable groups in class $\mathcal{S}$, and we prove a unique prime factorization result for group von Neumann algebras of products of groups in class $\mathcal{S}$.

Beknopte samenvatting

De groeps-von Neumannalgebra van een groep G wordt gedefinieerd als de sluiting van het lineaire opspansel van de linkse reguliere representatie $\{\lambda_g\}_{g \in G}$ in de zwakke operatortopologie. Op een gelijkaardige manier construeert men een von Neumannalgebra $L^\infty(X) \rtimes G$ uit elke niet-singuliere actie $G \curvearrowright (X, \mu)$ via de groep-maatruimteconstructie van Murray en von Neumann. Een centraal probleem in het onderzoek naar von Neumannalgebra's is de classificatie van $L(G)$ en $L^\infty(X) \rtimes G$ in termen van respectievelijk de onderliggende groep G en de onderliggende actie $G \curvearrowright (X, \mu)$. Door Ozawa's klasse $\mathcal{S}$ te definiëren voor lokaal compacte groepen, bewijzen we de eerste rigiditeits- en classificatieresultaten voor groeps-von Neumannalgebra's en groep-maatruimte von Neumannalgebra's gegeven door lokaal compacte groepen.

Klasse $\mathcal{S}$ voor discrete groepen speelt een belangrijke rol in rigiditeitsresultaten voor groeps-von Neumannalgebra's gegeven door discrete groepen. We definiëren klasse $\mathcal{S}$ voor lokaal compacte groepen en we geven een karakterisatie van lokaal compacte groepen in deze klasse in termen van groepen met een uitmiddelbare actie op een compactificatie die klein is op oneindig. Hiermee veralgemenen we een stelling van Ozawa. We geven voorbeelden van lokaal compacte groepen in klasse $\mathcal{S}$ en we bewijzen dat klasse $\mathcal{S}$ gesloten is onder maatequivalentie.

We bewijzen dat voor elke vrije, ergodische, kansmaatbewarende actie van een licht uitmiddelbare groep in klasse $\mathcal{S}$ de groep-maatruimte von Neumannalgebra $L^\infty(X) \rtimes G$ een unieke Cartandeelalgebra heeft, op unitaire conjugatie na. We leiden hieruit een W*-sterke rigiditeitsstelling af voor irreducibele acties van producten van zo'n groepen. Deze stellingen zijn in het bijzonder van toepassing op samenhangende, enkelvoudige Liegroepen met reële rang één en eindig centrum en op groepen met een eigenlijke actie op een boom of een hyperbolische graaf. We bewijzen verder sterke soliditeitsresultaten voor groeps-von Neumannalgebra's van licht uitmiddelbare groepen in klasse $\mathcal{S}$ en we bewijzen unieke priemfactorisatie voor groeps-von Neumannalgebra's van producten van groepen in klasse $\mathcal{S}$.

Contents

Chapter 1

Introduction

In this chapter, we give a brief introduction to von Neumann algebras and the phenomenon of W*-rigidity. We present the main results of this thesis, which were obtained in the author's joint publication [BDV18] with Arnaud Brothier and Stefaan Vaes and in the author's publication [Dep19]. In particular, parts of this introduction originate from these two articles.

1.1 Von Neumann algebras

In a series of papers in the 1930s and 1940s, Murray and von Neumann [MvN36; MvN37; vNeu40; MvN43] introduced the notion of a *von Neumann algebra* (or *W^*-algebra*), back then called 'rings of operators', as a subalgebra of the algebra $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ that is closed under taking adjoints and that is closed in the w.o. topology. The two most important classes of examples of von Neumann algebras for this thesis are the group von Neumann algebras and the group measure space von Neumann algebras. Both constructions were already introduced in the above mentioned seminal papers of Murray and von Neumann.

Given a locally compact group G, the left regular representation $\lambda : G \to B\big(L^2(G)\big)$ is given by

$$(\lambda_g f)(t) = f(g^{-1}t)$$

for all $f \in L^2(G)$, $g \in G$ and almost every (a.e.) $t \in G$. The *group von Neumann algebra* $L(G)$ is then defined as the von Neumann algebra generated by the operators $\{\lambda_g\}_{g \in G}$.

Closely related to the previous construction is the group measure space von Neumann algebra, associated to a nonsingular group action. Given a nonsingular action $G \curvearrowright (X,\mu)$ of a locally compact group G on a measure space (X,μ), the *group measure space von Neumann algebra* $L^\infty(X) \rtimes G$ is a von Neumann algebra generated by a copy of $L^\infty(X)$ and a copy of the group G, in form of group of unitaries $\{u_g\}_{g\in G}$ satisfying $u_g u_h = u_{gh}$. The copy of $L^\infty(X)$ and of the group G are such that the relation $u_g^* f u_g = \alpha_g(f)$ for $g,h \in G$ and $f \in L^\infty(X)$ is satisfied. Here, α is given by

$$(\alpha_g f)(x) = f(g^{-1}x)$$

for $f \in L^\infty(X)$, $g \in G$ and a.e. $x \in X$.

The 'simple' objects among the von Neumann algebras are the so-called factors. A von Neumann algebra M is called a *factor* if it has trivial center, i.e. $\mathcal{Z}(M) = \mathbb{C}1$. In [vNeu49], von Neumann showed that every von Neumann algebra M can be written as a direct integral (a 'generalized direct sum') of factors. Murray and von Neumann furthermore divided factors into three types. A factors is of type I if it is isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. A factor is of type II, if it is not of type I, but still admits a (possibly infinite) trace $\tau : M^+ \to [0,+\infty]$. If this trace exists, it is unique up to scaling by a positive number. If $\tau(1) < +\infty$, then the factor is of type II_1 and if $\tau(1) = +\infty$, then the factor is of type II_∞. All remaining factors are of type III.

Apart from the trivial case of type I, the factors of type II_1 are the best understood. Even for these factors, though, many open questions remain. For instance, one of the major open problems is the *free group factor problem*: is $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ whenever $n \neq m$? Also other types of factors exhibit interesting behavior, and their study has been made accessible through the type II_1 case by work of Connes, Tomita and Takesaki [Con73; Tom67; Tak70; Tak73; CT77].

The group von Neumann algebra $L(\Gamma)$ of a discrete group Γ is a factor if and only if Γ has *infinite conjugacy classes (icc)*, meaning that every conjugacy class except $\{e\}$ is infinite. In that case $L(\Gamma)$ is always of type II_1. For the group von Neumann algebra $L(G)$ of a (nondiscrete) locally compact group G, such a nice characterization does not exist. Moreover, for many such groups $L(G)$ is type I. However, examples of groups G such that $L(G)$ is a non-type I factor were provided by Godement [God51], Sutherland [Sut78].

The group measure space von Neumann algebra $L^\infty(X) \rtimes G$ is a factor if the action $G \curvearrowright (X,\mu)$ is ergodic and (essentially) free. Recall that an action $G \curvearrowright (X,\mu)$ is said to be *ergodic* if the only G-invariant sets $E \subseteq X$ are either null of conull. An action $G \curvearrowright (X,\mu)$ is said to be *essentially free* if the set $\{x \in X \mid \exists g \in G \setminus \{e\} : gx = x\}$ is a null set. If G is discrete and the action

$G \curvearrowright (X, \mu)$ is *probability measure preserving (pmp)*, then $L^\infty(X) \rtimes G$ is of type II_1.

Assumptions In this thesis, all von Neumann algebras are assumed to have a faithful representation on a separable Hilbert space. All groups G are assumed to be locally compact and second countable. All measure spaces (X, μ) are assumed to be standard measure spaces and all actions $G \curvearrowright (X, \mu)$ are assumed to be measurable.

1.2 W*-rigidity

It is natural to ask what the two constructions above 'remember' about the group G and the action $G \curvearrowright (X, \mu)$. For amenable groups G, it follows from the groundbreaking result of Connes in [Con76] that all group factors $L(G)$ and all group measure space factors $L^\infty(X) \rtimes G$ are completely classified by their type and (in the type III case) their flow of weights. In particular, $L(\Gamma)$ and $L^\infty(X) \rtimes \Gamma$ are all isomorphic for all infinite, amenable, discrete groups Γ and all free, ergodic, probability measure preserving (pmp) actions $\Gamma \curvearrowright (X, \mu)$. This means that within the class of amenable groups, almost all information about the group is lost when passing to the level of von Neumann algebras.

In general, more information is preserved for nonamenable groups. Some structural properties of nonamenable groups can be recovered by only looking at the associated von Neumann algebras. For instance, property (T), weak amenability and the Haagerup property for a countable group Γ can be recovered from the associated group von Neumann algebra (see [CJ85; CH89; Con82; Cho83]). Such a phenomenon is called *W*-rigidity*. In the last two decades Popa's deformation/rigidity theory, developed in [Pop06a; Pop06b; Pop06c], has lead to a wealth of such rigidity theorems for group von Neumann algebras and group measure space von Neumann algebras for countable groups (see [Pop07; Vae11; Ioa13; Vae16; Ioa17] for surveys). For instance, building on work of Ozawa and Popa [OP10b] and Gaboriau [Gab00], Popa and Vaes [PV14a] were able to show that $L^\infty(X) \rtimes \mathbb{F}_n$ and $L^\infty(Y) \rtimes \mathbb{F}_m$ are not isomorphic whenever $n \neq m$ and the actions $\mathbb{F}_n \curvearrowright (X, \mu)$ and $\mathbb{F}_m \curvearrowright (Y, \nu)$ are free, ergodic and pmp. This solved the dynamical version of the free group factor problem mentioned above. In [Pet10], Peterson even proved the existence of an action that is completely remembered by the von Neumann algebra. More precisely, an action $\Gamma \curvearrowright (X, \mu)$ is said to be *W*-superrigid* if $L^\infty(X, \mu) \rtimes \Gamma \cong L^\infty(Y, \nu) \rtimes \Lambda$ for any other action $\Lambda \curvearrowright (Y, \nu)$ implies that $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are conjugate: there exists a group isomorphism $\varphi : \Gamma \to \Lambda$ and a Borel isomorphism $\theta : X \to Y$

such that $\theta(gx) = \varphi(g)\theta(x)$ for a.e. $x \in X$ and all $g \in \Gamma$. Since then, more concrete examples of W*-superrigid actions have been discovered in among others [PV10; HPV13; Ioa11; GIT16].

On the level of the group von Neumann algebras, similar striking results have been obtained. A von Neumann algebra M is called *solid* if for every diffuse von Neumann subalgebra $A \subseteq L(\Gamma)$ that is the range of a normal, faithful conditional expectation, the relative commutant $A' \cap L(\Gamma)$ is amenable. Recall that a von Neumann algebra is called diffuse if it does not contain any minimal projections. Ozawa showed in [Oza04] that $L(\Gamma)$ is *solid* whenever Γ is a countable group in class $\mathcal{S}$ (see below for a definition). In particular, if Γ is nonamenable, icc and in class $\mathcal{S}$, then the group von Neumann algebra $L(\Gamma)$ is a *prime* factor, i.e. it does not decompose as a tensor product $M_1 \otimes M_2$ of non-type I factors M_1 and M_2.

In [OP04], Ozawa and Popa proved the first unique prime factorization results with groups in class $\mathcal{S}$. Among other results, they proved that if $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a product of nonamenable, icc groups in class $\mathcal{S}$, then $L(\Gamma) \cong L(\Gamma_1) \mathbin{\overline{\otimes}} \dots \mathbin{\overline{\otimes}} L(\Gamma_n)$ remembers the number of factors n and each factor $L(\Gamma_i)$ up to amplification, i.e. if $L(\Gamma) \cong N_1 \mathbin{\overline{\otimes}} \dots \mathbin{\overline{\otimes}} N_m$ for some prime factors $N_1, \dots, N_m$, then $n = m$ and (after relabeling) $L(\Gamma_i)$ is stably isomorphic to N_i for $i = 1, \dots, n$.

A stronger indecomposability property, called *strong solidity*, was introduced by Ozawa and Popa in [OP10b]. A von Neumann algebra M is said to be strongly solid if for every diffuse, amenable von Neumann subalgebra $A \subseteq M$ that is the range of a normal, faithful conditional expectation, the von Neumann subalgebra generated by the normalizer $\mathcal{N}_M(A) = \{x \in \mathcal{U}(M) \mid uAu^* = A\}$ remains amenable. Strong solidity of $L(\mathbb{F}_n)$ was established in [OP10b] and, more generally, for $L(\Gamma)$ with Γ weakly amenable and in class $\mathcal{S}$ by Chifan and Sinclair in [CS13]. The first strong solidity result for type III factors was obtained by Boutonnet, Houdayer, and Vaes in [BHV18].

In the W*-rigidity results for the group measure space construction mentioned above, the subalgebra $L^\infty(X)$ inside $M = L^\infty(X) \rtimes \Gamma$ plays a crucial role. By Singer's theorem from [Sin55], there exists an isomorphism $\pi : L^\infty(X) \rtimes \Gamma \to L^\infty(Y) \rtimes \Lambda$ such that $\pi\big(L^\infty(X)\big) = L^\infty(Y)$ if and only if the associated *orbit equivalence relations* $\mathcal{R}(\Gamma \curvearrowright X)$ and $\mathcal{R}(\Lambda \curvearrowright Y)$ are isomorphic, where

$$\mathcal{R}(\Gamma \curvearrowright X) = \{(gx, x) \mid g \in \Gamma, x \in X\}.$$

The subalgebra $A = L^\infty(X)$ is a *Cartan subalgebra*: it is maximal abelian, the normalizer $\mathcal{N}_M(A)$ generates M and there exists a normal, faithful conditional expectation $E : M \to A$. Hence, if one can show that the subalgebra A is the unique Cartan subalgebra of M (up to conjugation by an automorphism), then

Singer's theorem implies that $L^\infty(X) \rtimes \Gamma$ is isomorphic to $L^\infty(Y) \rtimes \Lambda$ if and only if $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$. The first such *Cartan rigidity* result was obtained by Ozawa and Popa in [OP10b]. They proved that $M = L^\infty(X) \rtimes \mathbb{F}_n$ has a unique Cartan subalgebra *up to unitary conjugacy* for *profinite*, free, ergodic, pmp actions $\mathbb{F}_n \curvearrowright (X, \mu)$. Here, we say that two Cartan subalgebras $A, B \subseteq M$ are unitarily conjugate if there exists a unitary $u \in \mathcal{U}(M)$ such that $B = uAu^*$. This result was later generalized to profinite, free, ergodic, pmp actions of weakly amenable groups in class $\mathcal{S}$ by Chifan and Sinclair in [CS13]. In [PV14a] and [PV14b], Popa and Vaes obtained the same result for *arbitrary* free, ergodic, pmp actions of $\mathbb{F}_n$ and of weakly amenable groups in class $\mathcal{S}$, respectively. Since then many other Cartan rigidity results were obtained in among others [CP13; CSU13; Ioa15; HV13; BHR14]. The first uniqueness theorem for Cartan subalgebras in type III factors was proven by Houdayer and Vaes in [HV13].

W*-rigidity results are then obtained by combining such Cartan rigidity results with *orbit equivalence (OE) rigidity* results, allowing one to obtain information about an action $\Gamma \curvearrowright (X, \mu)$ from the orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$. The non-isomorphism result $L^\infty(X) \rtimes \mathbb{F}_n \not\cong L^\infty(Y) \rtimes \mathbb{F}_m$ mentioned above was for instance obtained by combining the unique Cartan result from [PV14a] with an OE rigidity result from Gaboriau in [Gab00] stating that the orbit equivalence relations $\mathcal{R}(\mathbb{F}_n \curvearrowright X)$ and $\mathcal{R}(\mathbb{F}_m \curvearrowright Y)$ are not isomorphic whenever $m \neq n$ and $\mathbb{F}_n \curvearrowright (X, \mu)$ and $\mathbb{F}_m \curvearrowright (Y, \nu)$ are free, ergodic and pmp.

As we will explain below, together with Brothier and Vaes, the author was able to obtain the first W*-rigidity results for von Neumann algebras associated to general locally compact groups.

1.3 The class $\mathcal{S}$

As illustrated above, *class* $\mathcal{S}$ is an important class of groups for W*-rigidity phenomena. Class $\mathcal{S}$ for countable groups was introduced by Ozawa in [Oza06] as the class of countable groups Γ that are exact and that admit a map $\eta : \Gamma \to \text{Prob}(\Gamma)$ satisfying

$$\lim_{k \to \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

for all $g, h \in \Gamma$. Here, $\|.\|$ denotes the norm of total variation. Recall that a group Γ is called *exact* if the operation of taking the reduced crossed product preserves short exact sequences, or equivalently, if Γ admits a topologically amenable action on a compact space (see [Oza00; Ana02; BCL17]). By [Oza06, Theorem 4.1], class $\mathcal{S}$ can be characterized as the class of all groups that admit a topologically amenable action on a boundary that is *small at infinity*. (See Section 2.1.4 below for a definition of topological amenability.) Groups in class $\mathcal{S}$ are also called *bi-exact*.

Examples of countable groups in class $\mathcal{S}$ are amenable groups, nonabelian free groups $\mathbb{F}_n$, hyperbolic groups (see [Ada94]), lattices in connected simple Lie groups of real rank one with finite center (see [Ska88, Proof of Théorème 4.4]), wreath products $B \wr \Gamma$ with B amenable and Γ in class $\mathcal{S}$ (see [Oza06]) and $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ (see [Oza09]). Moreover, class $\mathcal{S}$ is closed under measure equivalence (see [Sak09a]). Examples of groups not belonging to class $\mathcal{S}$ are nonamenable inner amenable groups, nonamenable groups with infinite center and product groups $\Gamma \times \Lambda$ with Γ nonamenable and Λ infinite.

In Chapter 3, which is mostly based on the author's publication [Dep19], we define and study class $\mathcal{S}$ for locally compact groups, we prove a characterization similar to [Oza06, Theorem 4.1], we provide examples of groups in this class and we prove that class $\mathcal{S}$ is closed under measure equivalence.

Given a locally compact group G, we denote by $\mathrm{Prob}(G)$ the space of all Borel probability measures on G. This space can also be viewed as the state space of $C_0(G)$. We equip $\mathrm{Prob}(G)$ with the norm of total variation. The precise definition of class $\mathcal{S}$ for locally compact groups is now as follows.

Definition A. Let G be a locally compact group. We say that G belongs to *class $\mathcal{S}$* (or is *bi-exact*) if G is exact and if there exists a $\|.\|$-continuous map $\eta : G \to \mathrm{Prob}(G)$ satisfying

$$\lim_{k\to\infty} \|\eta(gkh) - g\cdot\eta(k)\| = 0 \tag{1.3.1}$$

uniformly on compact sets for $g, h \in G$.

Our first main result is a version of [Oza06, Theorem 4.1] for locally compact groups in class $\mathcal{S}$, i.e. we characterize groups in class $\mathcal{S}$ as groups acting amenably on a boundary that is *small at infinity*. Given a locally compact group G, we denote by $C_b^u(G)$ the algebra of *bounded uniformly continuous* functions on G, i.e. the bounded functions $f : G \to \mathbb{C}$ such that

$$\|\lambda_g f - f\|_\infty \to 0 \qquad \text{and} \qquad \|\rho_g f - f\|_\infty \to 0$$

whenever $g \to e$. Here, λ and ρ denote the left and right regular representations respectively defined by $(\lambda_g f)(h) = f(g^{-1}h)$ and $(\rho_g f)(h) = f(hg)$ for $f \in C_b(G)$ and $g, h \in G$. We define the compactification h^uG of the group G as the spectrum of the algebra

$$C(h^uG) \cong \{f \in C_b^u(G) \mid \rho_g f - f \in C_0(G) \text{ for all } g \in G\}$$

and denote by $\nu^u G = h^u G \setminus G$ its boundary. The compactification $h^u G$ is equivariant in the sense that both actions $G \curvearrowright G$ by left and right translation extend to continuous actions $G \curvearrowright h^u G$. It is also *small at infinity* in the sense that the extension of the action by right translation is trivial on the boundary $\nu^u G$.

The characterization of groups in class $\mathcal{S}$ now goes as follows.

Theorem B. *Let G be a locally compact group. Then, the following are equivalent*

- *(i)* *G is in class $\mathcal{S}$,*
- *(ii)* *the action $G \curvearrowright \nu^u G$ induced by left translation is topologically amenable,*
- *(iii)* *the action $G \curvearrowright h^u G$ induced by left translation is topologically amenable,*
- *(iv)* *the action of $G \times G$ on the spectrum of $C_b^u(G)/C_0(G)$ induced by left and right translation is topologically amenable.*

The two novelties in this result are condition (iii) and the method we used to prove the implication (iv)⇒(i).

Examples of locally compact groups in class $\mathcal{S}$ include amenable groups and connected simple Lie groups of real rank one with finite center (see [Ska88, Proof of Théorème 4.4]). Easy examples of groups not belonging to class $\mathcal{S}$ include product groups $G \times H$ with G nonamenable and H noncompact (see Proposition 3.4.6), nonamenable groups G with noncompact center (see Proposition 3.4.7) and nonamenable groups G that are *inner amenable at infinity*, i.e. for which there exists a conjugation invariant mean m on G such that $m(E) = 0$ for every compact set $E \subseteq G$ (see also Proposition 3.4.8).

We provide the following examples of groups in class $\mathcal{S}$. The first class of examples was obtained in the author's joint publication with Brothier and Vaes [BDV18]. In the case of countable groups, this result was proven by [Ada94].

Proposition C. *Let G be a locally compact group. If one of the two following conditions is satisfied, then G belongs to class $\mathcal{S}$.*

- *(i)* *G admits a continuous action on a (not necessarily locally finite) tree that is metrically proper in the sense that for every vertex x, we have that $d(x, g \cdot x) \to \infty$ when g tends to infinity in G.*

(ii) *G admits a continuous, proper action on a hyperbolic graph with uniformly bounded degree.*

Cornulier introduced the appropriate notion of wreath products for locally compact groups in [Cor17]. See (3.4.9) on page 130 for a definition of this notion along with the notation used here. Using Theorem B, we prove that certain of such locally compact wreath products are in class $\mathcal{S}$. This result is a locally compact version of [Oza06, Corollary 4.5].

Theorem D. *Let B and H be locally compact groups, X a countable set with a continuous action $H \curvearrowright X$ and $A \subseteq B$ be a compact open subgroup. If B is amenable, all stabilizers $\mathrm{Stab}_H(x)$ for $x \in X$ are amenable and H belongs to class $\mathcal{S}$, then also the wreath product $B \wr_X^A H$ belongs to class $\mathcal{S}$.*

A notion of measure equivalence for locally compact groups was introduced by S. Deprez and Li in [DL14]. We will recall this notion in Section 2.6. We proved the following result.

Theorem E. *The class $\mathcal{S}$ is closed under measure equivalence.*

For countable groups this result was proven by Sako in [Sak09a]. By [DL15, Corollary 2.9] and [DL14, Theorem 0.1 (6)] exactness is preserved under this notion of measure equivalence. To prove that the second condition (i.e. the existence of a map $\eta : G \to \mathrm{Prob}(G)$ as in (1.3.1)) is also preserved under measure equivalence, we use the characterization of measure equivalence in terms of isomorphic cross section equivalence relations from [KKR17; KKR18].

As a consequence of this theorem, we have for instance that $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ belong to class $\mathcal{S}$. Indeed, $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a lattice in both $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. Hence, the latter two are measure equivalent with $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$, which belongs to class $\mathcal{S}$ by [Oza09].

1.4 W*-rigidity for locally compact groups

In Chapter 4, we prove W*-rigidity results for locally compact groups. Chapter 4 is mainly based on the author's joint publication [BDV18] with Arnaud Brothier and Stefaan Vaes. We prove the following uniqueness theorem for Cartan subalgebras in the group measure space construction of locally compact groups. In Section 4.3, we actually prove a more general result, also valid for nonsingular actions (see Theorem 4.3.1) and thus generalizing the results in [HV13] to the locally compact setting.

Theorem F. *Let $G = G_1 \times \cdots \times G_n$ be a direct product of nonamenable, weakly amenable locally compact groups in class $\mathcal{S}$. If $G \curvearrowright (X, \mu)$ is an essentially free, pmp action, then $M = L^\infty(X) \rtimes G$ has a unique Cartan subalgebra up to unitary conjugacy.*

To understand Theorem F, note that if G is nondiscrete, then $L^\infty(X)$ is not a Cartan subalgebra of M since there is no faithful, normal conditional expectation $E : M \to L^\infty(X)$. However, M still contains a canonical Cartan subalgebra given by choosing a cross section for $G \curvearrowright (X, \mu)$ (see Section 2.5.5).

Examples of groups satisfying the conditions of Theorem F are the following. Every finite center connected simple Lie group G of real rank one with finite center is weakly amenable by [CH89]. Every locally compact group acting continuously and properly on a locally finite hyperbolic graph is weakly amenable by [Oza07]. Every locally compact group G that acts metrically properly on a tree even has CMAP by [Szw91] (and thus is certainly weakly amenable). As mentioned above, all these groups are also in class $\mathcal{S}$.

From Theorem F, we deduce the following W*-strong rigidity theorem. This is the first W*-strong rigidity theorem for actions of locally compact groups. Recall here that a nonsingular action $G_1 \times G_2 \curvearrowright (X, \mu)$ of a direct product group is called irreducible if both G_1 and G_2 act ergodically.

Theorem G. *Let $G = G_1 \times G_2$ and $H = H_1 \times H_2$ be unimodular locally compact groups without nontrivial compact, normal subgroups. Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \eta)$ be essentially free, irreducible, pmp actions. Assume that G_1, G_2, H_1, H_2 are nonamenable and that H_1, H_2 are weakly amenable and in class $\mathcal{S}$.*

If $p(L^\infty(X) \rtimes G)p \cong q(L^\infty(Y) \rtimes H)q$ for nonzero projections p and q, then the actions are conjugate: there exists a continuous group isomorphism $\delta : G \to H$ and a pmp isomorphism $\Delta : X \to Y$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ for all $g \in G$ and a.e. $x \in X$.

This result is obtained by first proving a cocycle superrigidity theorem for arbitrary cocycles of irreducible pmp actions $G_1 \times G_2 \curvearrowright (X, \mu)$ taking values in a locally compact group in class $\mathcal{S}$ (see Theorem 4.4.1). This result is very similar to the cocycle superrigidity theorem of [MS04], where the target group is assumed to be a closed subgroup of the isometry group of a negatively curved space. We then deduce that Sako's orbit equivalence rigidity theorem [Sak09b] for irreducible pmp actions $G_1 \times G_2 \curvearrowright (X, \mu)$ of nonamenable groups in class $\mathcal{S}$ stays valid in the locally compact setting (see Theorem 4.4.2). Combining these results with Theorem F then yields Theorem G.

We also obtain results on group von Neumann algebras of locally compact groups. We prove that the group von Neumann algebra of locally compact groups in class $\mathcal{S}$ are solid (see Proposition 4.6.1), generalizing [Oza04] to the locally compact setting. Using the same techniques as in proving Theorem F, we obtain the following strong solidity theorem for group von Neumann algebras of locally compact groups. Following [BHV18], we call a von Neumann algebra M *stably strongly solid* if the amplification $B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} M$ is strongly solid.

Theorem H. *Let G be a locally compact group in class $\mathcal{S}$ and assume that $L(G)$ is diffuse.*

1. *If G is unimodular and weakly amenable, then for every finite trace projection $p \in L(G)$, we have that $pL(G)p$ is strongly solid.*
2. *If G has the complete metric approximation property (CMAP) and if the kernel of the modular function $G_0 = \{g \in G \mid \delta(g) = 1\}$ is an open subgroup of G, then $L(G)$ is stably strongly solid.*

Note that the von Neumann algebras $L(G)$ appearing in the second part of Theorem H can be of type III.

We deduce Theorems F and H from very general structural results on the normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ and the stable normalizer $\mathcal{N}^{\mathrm{s}}_M(A) = \{x \in M \mid xAx^* \subseteq A \text{ and } x^*Ax \subseteq A\}$ of a von Neumann subalgebra $A \subset M$ when M is equipped with an arbitrary *coaction* $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ of a locally compact, weakly amenable group G in class $\mathcal{S}$, see Theorems 4.2.2 and 4.5.1. The main novelty is to show that the approaches of [PV14a] and [BHV18] remain applicable in this very general and much more abstract setting.

Theorem F is then obtained by applying these general results to the canonical coaction $\Phi : L(\mathcal{R}) \to L(\mathcal{R}) \mathbin{\overline{\otimes}} L(G)$ associated with a countable pmp equivalence relation $\mathcal{R}$ and a cocycle $\omega : \mathcal{R} \to G$ with values in the locally compact group G (see Section 4.1.1). Applying the same general results to the comultiplication $\Delta : L(G) \to L(G) \mathbin{\overline{\otimes}} L(G)$ itself, we obtain Theorem H.

From the solidity result mentioned above, it follows that whenever G is a locally compact group in class $\mathcal{S}$ such that $L(G)$ is also a nonamenable factor, then $L(G)$ is prime. Combining Theorem B with the unique prime factorization results of Houdayer and Isono in [HI17] along with the generalization [AHHM18, Application 4] by Ando, Haagerup, Houdayer, and Marrakchi, the author was able to obtain the following unique prime factorization result for (tensor products of) such group von Neumann algebras in [Dep19].

Theorem I. *Let $G = G_1 \times \cdots \times G_m$ be a direct product of locally compact groups in class $\mathcal{S}$ whose group von Neumann algebras $L(G_i)$ are nonamenable* *factors. If*

$$L(G) \cong N_1 \mathbin{\overline{\otimes}} \ldots \mathbin{\overline{\otimes}} N_n$$

for some non-type I factors N_i, then $n \leqslant m$. Moreover, all factors N_i are prime if and only if $n = m$ and in that case (after relabeling) $L(G_i)$ is stably isomorphic to N_i for $i = 1, \ldots, n$.

We prove this theorem by proving that for groups G in class $\mathcal{S}$, the group von Neumann algebra $L(G)$ belongs to the class $\mathcal{C}_{(AO)}$ introduced in [HI17].

It is worthwhile to note that for many locally compact groups G, the group von Neumann algebra $L(G)$ is amenable or even type I. For instance, the group von Neumann algebra of a connected locally compact group is always amenable by [Con76, Corollary 6.9] and $L\big(\mathrm{SL}_n(\mathbb{Q}_p)\big)$ is type I by [Ber74]. However on page 140, we present an example due to Suzuki of a group whose group von Neumann algebra $L(G)$ is a nonamenable factor of type II_∞. Furthermore, certain classes of groups acting on trees have nonamenable group von Neumann algebras by [HR19, Theorem C and D]. Also, [Rau19b, Theorem E and F] would provide conditions on such groups under which $L(G)$ is a nonamenable factor. In particular, for every $q \in \mathbb{Q}$ with $0 < q < 1$ [Rau19b, Theorem G] would provide examples of groups in class $\mathcal{S}$ for which the group von Neumann algebra is a nonamenable factor of type III_q. However, due to an error in [Rau19b, Lemma 5.1] in that paper, there is a gap in the proofs of these results (see also [Rau19a, p 20]), and it is currently not completely clear whether these results hold as stated there.

Chapter 2

Preliminaries

In this chapter, we will recall some notions from (geometric) group theory, operator algebras and dynamical systems that we use throughout the rest of this thesis. We will also prove some small results that we will need later in this thesis and for which no reference is available in the literature.

We assume that the reader is familiar with the basic ideas and concepts of measure theory, Hilbert spaces and Banach spaces. Given a Hilbert space $\mathcal{H}$, we denote by $B(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. Recall that apart from the norm topology, $B(\mathcal{H})$ can be equipped with the following three weaker topologies.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space. On $B(\mathcal{H})$, we define

(a) the *weak operator (w.o.) topology* as the weakest topology for which the maps $T \mapsto \langle T\xi, \eta \rangle$ are continuous for all $\xi, \eta \in \mathcal{H}$,

(b) the *strong operator (s.o.) topology* as the weakest topology for which the maps $T \mapsto \|T\xi\|$ are continuous for all $\xi \in \mathcal{H}$,

(c) the *strong* operator topology* as the weakest topology for which the maps $T \mapsto \|T\xi\|$ and $T \mapsto \|T^*\xi\|$ are continuous for all $\xi \in \mathcal{H}$.

2.1 Amenability and locally compact groups

Amenability is an approximation property of groups whose introduction in the first half of the 20^{th} was motivated by the theory of Lebesgue integration and in particular the Banach-Tarski paradox. It has since then become a central notion in geometric group theory and has found many applications in other areas of mathematics.

We will start this section by discussing the by now classical notion of *amenability* for locally compact groups, followed by the similar, but weaker notions of *inner amenability* and *weak amenability*. We end the section by discussing *topological amenability*, a notion of amenability of continuous actions of locally compact groups.

Throughout this thesis, we will typically denote (not necessarily discrete) locally compact groups by G, H and K, while countable groups are denoted by Γ and Λ. We will assume all locally compact groups to be second countable, unless stated otherwise. We denote by λ_G the left invariant Haar measure. The modular function is denoted by δ_G.

For $1 \leqslant p \leqslant \infty$, we denote by $\lambda, \rho : G \to B\big(L^p(G)\big)$ the *left* and *right regular representation* defined by

$$(\lambda_g f)(h) = f(g^{-1}h), \qquad (\rho_g f)(h) = f(hg)\delta_G(g)^{1/p} \tag{2.1.1}$$

for $f \in L^p(G)$, $g \in G$ and a.e. $h \in G$. Here, we used the convention that $1/\infty = 0$. Similarly, we denote by $\alpha : G \to B\big(L^p(G)\big)$ the *adjoint representation* defined by

$$(\alpha_g f)(h) = (\lambda_g \rho_g f)(h) = f(g^{-1}hg)\delta_G(g)^{1/p} \tag{2.1.2}$$

for $f \in L^p(G)$, $g \in G$ and a.e. $h \in G$.

Denote by $C_b(G)$ the algebra of bounded continuous functions $f : G \to \mathbb{C}$. We say that a function $f \in C_b(G)$ is *left (resp. right) uniformly continuous* if $\|\lambda_g f - f\|_\infty \to 0$ (resp. $\|\rho_g f - f\|_\infty \to 0$) whenever $g \to e$. We say that f is *uniformly continuous* if it is both left and right uniformly continuous. We denote by $C_b^{lu}(G)$ (resp. $C_b^{ru}(G)$) the algebra of left (resp. right) uniformly continuous functions on G and by $C_b^u(G)$ the algebra of uniformly continuous functions on G. We denote by $C_0(G)$ the algebra of functions converging to zero at infinity. Note that all these algebras can be viewed as subalgebras of $L^\infty(G)$.

The dual space of $C_0(G)$ is the space $M(G)$ of all complex Radon measures on G. We can equip this space with the weak* topology, or with topology coming from the norm $\|.\|$ of total variation. The Borel structure from both topologies agree. We denote by $M(G)^+$ the space of positive Radon measures and $\mathrm{Prob}(G)$ the space of Radon probability measures. For $g \in G$ and $\mu \in M(G)$, we denote by $g \cdot \mu$ the measure defined by $(g \cdot \mu)(E) = \mu(g^{-1}E)$ for all Borel sets $E \subseteq G$.

2.1.1 Amenability

The definition of amenability we present here is Day's generalization in [Day57] of von Neumann's original definition from [vNeu29]. Recent treatments of amenability for locally compact groups can be found in [Pat88], [Run02, Chapter 1] and [BdlHV08, Appendix G].

A *mean* on a group G is a linear functional $m : L^\infty(G) \to \mathbb{C}$ satisfying $m(1) = 1$ and $m(f) \geqslant 0$ for all $f \in L^\infty(G)$ with $f \geqslant 0$. This means that in the language of Section 2.3, a mean on G is a state on $L^\infty(G)$. A linear functional $m : E \to \mathbb{C}$ with $E = C_b(G), C_b^{lu}(G), C_b^{ru}(G), C_b^u(G), C_0(G)$ satisfying the same properties will be called a mean on E. On a countable group Γ every mean defines a finitely additive probability measure on G and vice versa.

Definition 2.1.1. Let G be a locally compact group. We say that G is *amenable* if it admits a *left invariant mean*, i.e. a mean $m : L^\infty(G) \to \mathbb{C}$ satisfying $m(\lambda_g f) = f$ for every $f \in L^\infty(G)$.

Of course an amenable group also admits a right invariant mean. One can even prove that every amenable group admits a mean that is both left and right invariant (see [Run02, Theorem 1.1.11]).

Obviously, every compact group is amenable, where the mean is given by integrating with respect to the (finite) Haar measure. Slightly less trivial examples are abelian groups (see [BdlHV08, Theorem G.2.1]), which is typically proved using the Markov-Kakutani fixed point theorem [Mar36; Kak38]. From the permanence properties below, it then also follows that all solvable groups are amenable. Examples of a nonamenable groups are $\mathbb{F}_n$ for $n \geqslant 2$ [vNeu29], i.e. the nonabelian free groups on $n \geqslant 2$ generators.

Recall that a continuous function $\varphi : G \to \mathbb{C}$ is called *positive definite* if for all $g_1, \ldots, g_n \in G$ the matrix $\big(\varphi(g_i^{-1} g_j)\big)_{i,j=1,\ldots,n}$ is positive semidefinite, i.e. for all $z_1, \ldots, z_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{n} \overline{z_i} z_j \varphi(g_i^{-1} g_j) \geqslant 0.$$

Amenability has a lot of equivalent characterizations. In the theorem below we mention a few. Proofs can be found in [Run02, Theorem 1.1.9 and Exercise 1.1.6] and [BdlHV08, Theorem G.3.1, Theorem G.3.2, Theorem G.5.1 and Theorem F.1.4]. In the formulation below, we denote by $\mathcal{S}(G)$ the space

$$\mathcal{S}(G) = \left\{ f \in L^1(G)^+ \;\middle|\; \|f\|_1 = 1 \right\}.$$

Theorem 2.1.2. *Let G be a locally compact group. Then, the following are equivalent.*

(i) *G is amenable, i.e. there exists a left invariant mean on $L^\infty(G)$.*

(ii) *[Nam66] There exists a left invariant mean on one (and hence all) of $C_b(G)$, $C_b^u(G)$, $C_b^{lu}(G)$ and $C_b^{ru}(G)$.*

(iii) *(Følner's condition, [Føl55]) There exists a sequence $(E_n)_n$ of Borel subsets of G with strictly positive finite Haar measure such that*

$$\lim_n \frac{\lambda_G(gE_n \triangle E_n)}{\lambda_G(E_n)} = 0$$

uniformly on compact sets for $g \in G$.

(iv) *(Reiter's condition, [Rei65]) There exists a sequence $(f_n)_n$ in $\mathcal{S}(G)$ such that*

$$\lim_n \|f_n - \lambda_g f_n\|_1 = 0$$

uniformly on compact sets for $g \in G$.

(v) *[Die60; Rei64] There exists a sequence $(\xi_n)_n$ in $L^2(G)$ with $\|\xi_n\|_2 = 1$ for all $n \in \mathbb{N}$ and such that*

$$\lim_n \|\xi_n - \lambda_g \xi_n\|_2 = 0$$

uniformly on compact sets for $g \in G$.

(vi) *There exists a sequence $(\mu_n)_n$ in $\mathrm{Prob}(G)$ such that*

$$\lim_n \|\mu_n - g \cdot \mu_n\| = 0$$

uniformly on compact sets for $g \in G$.

(vii) *[Rei64] There exists a sequence $(\varphi_n)_n$ of compactly supported positive definite functions on G such that $\varphi_n \to 1$ pointwise.*

The following three permanence properties for amenability were already proven in von Neumann's original paper [vNeu29] in the case of countable groups. The locally compact case is due to Greenleaf in [Gre69, Theorems 2.3.1-2.3.4]. A more recent self-contained proof can be found in [BdlHV08, Proposition G.2.2],

Theorem 2.1.3. *Let G be a locally compact group.*

(a) *If G is amenable, then so are all its closed subgroups.*

(b) If $N \lhd G$ is a closed normal subgroup, then G is amenable if and only if both N and G/N are amenable.

(c) If $(G_n)_n$ is an increasing sequence of amenable closed subgroups in G and $\bigcup_{n\in\mathbb{N}} G_n$ is dense in G, then G is amenable.

It follows for instance that every group containing $\mathbb{F}_2$ as a closed subgroup is not amenable. The converse, known as von Neumann's problem, remained open for a long time, but in [Ols80] Ol'shanskii constructed a nonamenable group all with the property that all of its proper subgroups are cyclic and thus certainly does not contain $\mathbb{F}_2$ as a closed subgroup.

The class $\mathcal{E}G$ of *elementary amenable groups* is the smallest class of groups that contains all finite and all amenable groups and that is closed under taking subgroups, quotients, extensions by groups in $\mathcal{E}G$ and unions of increasing sequences in $\mathcal{E}G$. By the previous theorem, clearly all groups in $\mathcal{E}G$ are indeed amenable. The converse was open remained open until Grigorchuk constructed an amenable group that is not elementary amenable in [Gri85].

2.1.2 Weak amenability and CMAP

Weak amenability is a weaker notion of amenability introduced by Cowling and Haagerup in [CH89]. The *Fourier algebra* $A(G)$ is the space of coefficients of the left regular representation, i.e. the space of maps $u_{\xi,\eta} : G \to \mathbb{C} : g \mapsto \langle \lambda(g)\xi, \eta\rangle$ for $\xi, \eta \in L^2(G)$. One can prove that this space is closed under pointwise addition and multiplication, so that $A(G)$ is indeed an algebra. The algebra $A(G)$ is equipped with the norm given by

$$\|u\|_A = \inf_{\substack{\xi,\eta\in L^2(G)\\ u=u_{\xi,\eta}}} \|\xi\|_2 \|\eta\|_2$$

This norm turns $A(G)$ into a Banach algebra (see [Eym64, Chapitre 2 and 3]).

A function $\varphi : G \to \mathbb{C}$ for which the pointwise product $\varphi u \in A(G)$ for every $u \in A(G)$ is called a *Fourier multiplier*. Note that every Fourier multiplier is necessarily continuous. Clearly, every $u \in A(G)$ is a Fourier multiplier.

We are mainly interested in the subclass of so-called *completely bounded* multipliers, which can be defined as follows. The motivation for the terminology *completely bounded* comes from Theorem 2.4.33 below.

Definition 2.1.4. A Fourier multiplier $\varphi : G \to \mathbb{C}$ is said to be *completely bounded* if there exist bounded, continuous maps $\xi, \eta : G \to \mathcal{H}$ to a Hilbert space $\mathcal{H}$ such that

$$\varphi(h^{-1}g) = \langle \xi(g), \eta(h) \rangle \tag{2.1.3}$$

for all $g, h \in G$. Denoting $\|\xi\|_\infty = \sup_{g\in G} \|\xi(g)\|$, we define by $\|\varphi\|_{\mathrm{cb}}$ as the infimum of all values $\|\xi\|_\infty \|\eta\|_\infty$ for all bounded, continuous map $\xi, \eta : G \to \mathcal{H}$ for which (2.1.3) holds.

Every $u \in A(G)$ is a completely bounded Fourier multiplier and $\|u\|_{\mathrm{cb}} \leqslant \|u\|_A$ by [dCH85, Corollary 1.8]. Moreover, by the same result, every coefficient of a unitary representation of G is a completely bounded Fourier multiplier. In particular, by the GNS-representation of positive definite functions (see [BdlHV08, Theorem C.4.10]), all positive definite functions are also completely bounded Fourier multipliers.

Weak amenability is now defined as follows.

Definition 2.1.5. A locally compact group G is said to be *weakly amenable* if there exists a sequence $(\varphi_n)_n$ of compactly supported, completely bounded Fourier multipliers such that $L = \sup_n \|\varphi_n\|_{\mathrm{cb}} < +\infty$ and $\varphi_n \to 1$ uniformly on compact sets. The *Cowling-Haagerup constant* Λ_G of G is the infimum of L over all such sequences.

The group G has the *complete metric approximation property (CMAP)* if G is weakly amenable with Cowling-Haagerup constant $\Lambda_G \leqslant 1$.

Note that if G is weakly amenable, then we can take a sequence $(\varphi_n)_n$ of compactly supported, completely bounded Fourier multipliers such that

$$\Lambda_G = \limsup_n \|\varphi_n\|_{\mathrm{cb}} .$$

Cowling and Haagerup originally used the following definition. See [CH89, Proposition 1.1] for a proof that these two definitions are indeed equivalent.

Proposition 2.1.6. *A locally compact group G is weakly amenable if and only if there exists a sequence $(u_n)_n$ in $A(G)$ such that $L = \sup_n \|u_n\|_{cb} < +\infty$ and such that $u_n \to 1$ uniformly on compact sets. Moreover, the elements u_n can be assumed to be compactly supported. The Cowling-Haagerup constant Λ_G is the infimum of L over all such sequences.*

Using the characterization of amenability in terms of positive definite functions, it is clear that amenable groups have CMAP. Another class of groups with CMAP is the following. This result was proven by Szwarc in [Szw91].

Proposition 2.1.7. *Let G be a locally compact group admitting a continuous action on a (not necessarily locally finite) tree that is metrically proper in the sense that for every vertex x, we have that $d(x, g \cdot x) \to \infty$ when g tends to infinity in G. Then, G has CMAP.*

The following result was proven by Cowling and Haagerup in [CH89] for groups with finite center. The non-finite center case later was proven by Hansen in [Han90].

Theorem 2.1.8. *Any real rank one, connected simple Lie group is weakly amenable.*

Recall that the real rank of a (semisimple) matrix Lie group is the maximal dimension of an abelian subgroup which can be diagonalized over $\mathbb{R}$ (see for instance [Zim84, p 1]). For instance, $\mathrm{SL}(n,\mathbb{R})$ has real rank $n-1$ for $n \geqslant 2$. Other examples are the following.

Example 2.1.9. Fix $n, m \in \mathbb{N}$. Denote by $I_n \in M_n(\mathbb{C})$ the identity matrix. Define

$$I_{n,m} = \begin{pmatrix} -I_n & 0 \\ 0 & I_m \end{pmatrix} \qquad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and

$$K_{n,m} = \begin{pmatrix} -I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}.$$

The following groups are examples of simple Lie groups

$$\mathrm{SO}(n,m) = \left\{ A \in \mathrm{SL}_{n+m}(\mathbb{R}) \;\middle|\; A^T I_{p,q} A = I_{p,q} \right\},$$

$$\mathrm{SU}(n,m) = \{ A \in \mathrm{SL}_{n+m}(\mathbb{C}) \mid A^* I_{p,q} A = I_{p,q} \},$$

$$\mathrm{Sp}(n,m) = \left\{ A \in \mathrm{GL}_{2n+2m}(\mathbb{C}) \;\middle|\; A^T J_{n+m} A = J_{n+m} \text{ and } A^* K_{p,q} A = K_{p,q} \right\}.$$

The real rank of the three groups $\mathrm{Sp}(n,m)$, $\mathrm{SO}(n,m)$ and $\mathrm{SU}(n,m)$ is $\min\{m,n\}$ (see for instance [OV94, Table 4]). Hence, by the previous theorem, the groups $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$ and $\mathrm{Sp}(1,n)$ are weakly amenable for each $n \in \mathbb{N}$. Moreover, Cowling and Haagerup calculated that $\Lambda_G = 1$ for $G = \mathrm{SO}(1,n)$ or $G = \mathrm{SU}(1,n)$ and that $\Lambda_G = 2n-1$ for $G = \mathrm{Sp}(1,n)$.

Noncompact, connected, simple Lie groups of real rank at least 2 are not weakly amenable by [Haa16; Dor93; Dor96].

We end this section by mentioning the following permanence properties for weak amenability. Proofs can be found in [CH89, Proposition 1.3, Corollary 1.5], [Jol15, Theorem 1.5] and [DL14, Example 0.4].

Proposition 2.1.10. *Let G and H be locally compact groups.*

(a) If G is weakly amenable, then so are all closed subgroups K. In that case $\Lambda_K \leqslant \Lambda_G$.

(b) If $(G_n)_n$ is an increasing sequence of weakly amenable open subgroups in G such that $G = \bigcup_n G_n$, then G is weakly amenable and $\Lambda_G = \sup_n \Lambda_{G_n}$.

(c) The direct product $G \times H$ is weakly amenable if and only if both G and H are. In that case $\Lambda_{G\times H} = \Lambda_G \Lambda_H$.

(d) If $K \subseteq G$ is a compact normal subgroup, then G is weakly amenable if and only if G/K is. In that case $\Lambda_G = \Lambda_{G/K}$.

(e) If H is measure equivalent to G (see Section 2.6), then G is weakly amenable if and only if H is.

Contrary to the case of amenable groups, it is *not* true that quotients of weakly amenable groups are weakly amenable. Indeed, every finitely generated group is a quotient of a nonabelian free group, and nonabelian free groups are weakly amenable by [Haa78]. But, for instance, $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is not weakly amenable by [Haa16, Remark on p10].

2.1.3 Inner amenability

Inner amenability is another weaker notion of amenability, first introduced by Effros in [Eff75] for countable groups and by A. L. Paterson in [Pat88] for locally compact groups. Results in this section are mostly taken from [BdlH86] and [Pat88].

Unfortunately, the terminology used in the literature for the theory of inner amenability for general locally compact groups is not compatible with the terminology used for countable, discrete groups. Since we will be mostly working with nondiscrete groups in this thesis, we will stick to the terminology for locally compact groups.

Definition 2.1.11. Let G be a locally compact group. We say that

(a) G is *inner amenable* if it admits a *conjugation invariant mean*, i.e. a mean $m : L^\infty(G) \to \mathbb{C}$ satisfying $m(\alpha_g f) = m(f)$, where α is defined as in (2.1.2).

(b) G is *inner amenable at infinity* if it admits a conjugation invariant mean $m : L^\infty(G) \to \mathbb{C}$ satisfying $m(f) = 0$ for all compact subsets $f \in C_0(G)$.

Since amenable groups admit a mean that is both left and right invariant, it follows that all noncompact amenable groups are inner amenable at infinity (and hence inner amenable). Note that with this terminology, all countable, discrete groups are inner amenable, since the mean concentrated on the unit is conjugation invariant. For this reason, inner amenability for countable, discrete groups is typically defined as what we define as inner amenability at infinity or as the existence of a conjugation invariant mean m satisfying $m(\delta_e) = 0$. The latter two definitions coincide for countable, discrete groups with infinite conjugacy classes (icc).

Apart from amenable groups, easy examples of groups that are inner amenable (resp. inner amenable at infinity) are groups with open center (resp. noncompact open center). More generally, all [IN] groups are inner amenable. Recall that a group is called [IN] if it admits a compact conjugation invariant neighborhood. Examples of groups that are not inner amenable are nonamenable connected groups (see [LR87, Theorem 1]). An example of a group that is inner amenable, but not inner amenable at infinity is $\mathbb{F}_2$ (this follows from [MvN43, Lemma 6.2.2]).

The following characterizations are due to Losert and Rindler in [LR87, Proposition 1].

Proposition 2.1.12. *Let G be a locally compact group. Then, the following are equivalent*

(i) *G is inner amenable.*

(ii) *There exists a sequence $(f_n)_n$ in $\mathcal{S}(G)$ such that*

$$\lim_n \|f_n - \alpha_g f_n\|_1 = 0$$

uniformly on compact sets for $g \in G$.

(iii) *There exists a sequence $(\xi_n)_n$ in $L^2(G)$ with $\|\xi_n\| = 1$ for all $n \in \mathbb{N}$ and such that*

$$\lim_n \|\xi_n - \alpha_g \xi_n\|_2 = 0$$

uniformly on compact sets for $g \in G$.

Proposition 2.1.13. *Let G be a locally compact group. Then, the following are equivalent*

(i) *G is inner amenable at infinity.*

(ii) *There exists a sequence $(f_n)_n$ in $\mathcal{S}(G)$ such that $\|f_n 1_K\|_1 \to 0$ for all compact $K \subseteq G$ and*

$$\lim_n \|f_n - \alpha_g f_n\|_1 = 0$$

uniformly on compact sets for $g \in G$.

(iii) *There exists a sequence $(\xi_n)_n$ in $L^2(G)$ with $\|\xi_n\|_2 = 1$ for all $n \in \mathbb{N}$ such that $\|f_n 1_K\|_2 \to 0$ for all compact $K \subseteq G$ and*

$$\lim_n \|\xi_n - \alpha_g \xi_n\|_2 = 0$$

uniformly on compact sets for $g \in G$.

2.1.4 Topological amenability

Topological amenability is a notion of amenability for continuous actions that was introduced by Anantharaman-Delaroche in [Ana87]. It was inspired by a similar notion in a measurable context by Zimmer in [Zim78]. This notion was later generalized to groupoids by Anantharaman-Delaroche and Renault in [AR00]. The specialization to locally compact groups presented here can be found in [Ana02].

In this section and in the rest of the thesis, we will denote topological spaces by X or Y. We assume all topological spaces to be Hausdorff and locally compact. All actions $G \curvearrowright X$ are assumed to be continuous (unless stated otherwise).

Definition 2.1.14. Let G be a locally compact group, X a locally compact space and $G \curvearrowright X$ a continuous action. We say that $G \curvearrowright X$ is *(topologically) amenable* if there exists a net of weakly* continuous maps $\mu_i : X \to \mathrm{Prob}(G)$ satisfying

$$\lim_i \|g \cdot \mu_i(x) - \mu_i(gx)\| = 0 \tag{2.1.4}$$

uniformly on compact sets for $x \in X$ and $g \in G$.

It is clear that all actions of amenable groups are topologically amenable. Conversely, if $G \curvearrowright X$ is topologically amenable, then all stabilizers are amenable. In particular, if $G \curvearrowright X$ has a fixed point, then G is amenable.

Anantharaman-Delaroche proved the following in [Ana02, Proposition 2.2].

Proposition 2.1.15. *Let G be a locally compact group, X a locally compact space and $G \curvearrowright X$ a continuous action. Then, the following are equivalent.*

(i) $G \curvearrowright X$ *is amenable.*

(ii) *There exists a net $(f_i)_i$ in $C_c(X \times G)^+$ satisfying*

$$\lim_i \int_G f_i(x,s)\,\mathrm{d}s = 1$$

uniformly on compact sets for $x \in X$ and

$$\lim_i \int_G |f_i(x, g^{-1}s) - f_i(gx, s)|\,\mathrm{d}s = 0 \tag{2.1.5}$$

uniformly on compact sets for $x \in X$ and $g \in G$.

Remark 2.1.16. Obviously, when X is σ-compact, we can replace nets by sequences in the above definition and proposition.

Remark 2.1.17. One can check that if X is a compact space, then we can take a sequence $(f_n)_n$ in $C_c(X \times G)^+$ satisfying $\int_G f_n(x,s)\,\mathrm{d}s = 1$ for every $x \in X$ and every $n \in \mathbb{N}$ and such that (2.1.5) holds.

The following result shows that if X is a σ-compact space, then one can assume that the convergence in (2.1.4) is uniform on the whole space X, instead of only uniform on compact sets of X. This result was proven by the author in [Dep19].

Proposition 2.1.18. *Let G be a locally compact group, X a σ-compact space and $G \curvearrowright X$ a continuous action. The action $G \curvearrowright X$ is amenable if and only if there exists a sequence of weakly* continuous maps $\mu_n : X \to \mathrm{Prob}(G)$ satisfying*

$$\lim_{n\to\infty} \|g \cdot \mu_n(x) - \mu_n(gx)\| = 0$$

uniformly on $x \in X$ and uniformly on compact sets for $g \in G$.

Proof. Suppose that $G \curvearrowright X$ is amenable. Since G is σ-compact, it suffices to construct for every compact set $K \subseteq G$ and every $\varepsilon > 0$ a weakly* continuous map $\mu : X \to \mathrm{Prob}(G)$ satisfying

$$\|g \cdot \mu(x) - \mu(gx)\| < \varepsilon \tag{2.1.6}$$

for all $g \in K$ and all $x \in X$.

So, fix a compact set $K \subseteq G$ and an $\varepsilon > 0$. Without loss of generality, we can assume that K is symmetric. Take an increasing sequence $(L_n)_{n\geqslant 1}$ of compact subsets in X such that $X = \bigcup_n L_n$. Since X is locally compact, after inductively enlarging L_n, we can assume that $L_n \subseteq \mathrm{int}(L_{n+1})$ and $gL_n \subseteq L_{n+1}$ for every $g \in K$. Using the amenability of $G \curvearrowright X$, we can take a sequence of weakly* continuous maps $\nu_n : X \to \mathrm{Prob}(G)$ satisfying

$$\|g \cdot \nu_n(x) - \nu_n(gx)\| < 2^{-n}$$

for all $g \in K$, $x \in L_n$ and $n \in \mathbb{N} \setminus \{0\}$. Set $L_n = \emptyset$ for $n \leqslant 0$. Fix $n \geqslant 1$ such that $18/n < \varepsilon$ and take continuous functions $f_k : X \to [0,1]$ such that $f_k(x) = 1$ whenever $x \in L_k \setminus L_{k-n}$ and $f_k(x) = 0$ whenever $x \in L_{k-n-1}$ or $x \in X \setminus L_{k+1}$.

For every $x \in X$, we denote $|x| = \max\{k \in \mathbb{N} \mid x \notin L_k\}$. Set

$$\tilde{\mu}(x) = \sum_{k=0}^{\infty} f_k(x)\nu_k(x) = f_{|x|}(x)\nu_{|x|}(x) + f_{|x|+n+1}(x)\nu_{|x|+n+1}(x) + \sum_{k=|x|+1}^{|x|+n} \nu_k(x).$$

for $x \in X$ and define $\mu : X \to \mathrm{Prob}(G) : x \mapsto \tilde{\mu}(x)/\|\tilde{\mu}(x)\|$. Clearly, μ is weakly* continuous. To prove that μ satisfies (2.1.6), fix $x \in X$ and $g \in K$. Since $gL_k \subseteq L_{k+1}$ and $g^{-1}L_k \subseteq L_{k+1}$ for every $k \in \mathbb{N}$, we have $|x| - 1 \leqslant |gx| \leqslant |x| + 1$ and hence

$$\|g \cdot \tilde{\mu}(x) - \tilde{\mu}(gx)\| \leqslant 8 + \sum_{k=|x|+1}^{|x|+n} \|g \cdot \nu_k(x) - \nu_k(gx)\| \leqslant 9,$$

where we used that $g \in K$ and $x \in L_k$ for $k = |x| + 1, \ldots, |x| + n$. Hence,

$$\|g \cdot \mu(x) - \mu(gx)\| \leqslant \frac{2}{\|\tilde{\mu}(x)\|} \|g \cdot \tilde{\mu}(x) - \tilde{\mu}(gx)\| \leqslant \frac{18}{n} < \varepsilon$$

as was required. □

Given a locally compact space X, we denote by $M(X)$ the space of all complex Radon measures on X. We equip $M(X)$ with the weak* topology induced by identifying $M(X)$ with the dual space of the algebra $C_0(X)$ of continuous functions $f : X \to \mathbb{C}$ converging to zero at infinity. We denote by $M(X)^+$ and $\mathrm{Prob}(X)$ the subspaces of all positive Radon measures and all Radon probability measures respectively.

The following result can for instance be found in [BO08, Exercise 15.2.1] in the case of discrete groups. For completeness, we include a proof for locally compact groups here.

Lemma 2.1.19. *Let G be a locally compact group, X a locally compact space and $G \curvearrowright X$ a continuous action. Then, $G \curvearrowright X$ is amenable if and only if the induced action $G \curvearrowright \mathrm{Prob}(X)$ is amenable, where $\mathrm{Prob}(X)$ is equipped with the weak* topology.*

Proof. Since the map $X \to \mathrm{Prob}(X) : x \mapsto \delta_x$ is weakly* continuous and G-equivariant, amenability of $G \curvearrowright \mathrm{Prob}(X)$ implies amenability of $G \curvearrowright X$.

Conversely, suppose that $G \curvearrowright X$ is amenable. Let $\eta_i : X \to \mathrm{Prob}(G)$ be a net of maps as in the definition. Then,

$$\tilde{\eta}_i : \mathrm{Prob}(X) \to \mathrm{Prob}(G) : \mu \mapsto \int_X \eta_i(x)\,\mathrm{d}\mu(x)$$

is weakly* continuous and satisfies

$$\begin{aligned}\|\tilde{\eta}_i(g \cdot \mu) - g \cdot \tilde{\eta}_i(\mu)\| &\leqslant \int_X \|\eta_i(gx) - g \cdot \eta_i(x)\|\,\mathrm{d}\mu(x) \\ &\leqslant \sup_{x \in X} \|\eta_i(gx) - g \cdot \eta_i(x)\| \to 0\end{aligned}$$

uniformly for $\mu \in \mathrm{Prob}(X)$ and uniformly on compact sets for $g \in G$. □

2.2 Hyperbolic metric spaces groups

Hyperbolicity is a robust 'large scale' negative curvature condition for metric spaces introduced by Gromov in [Gro87]. The most general definition of hyperbolic metric spaces is the following.

Definition 2.2.1. A metric space (X, d) is *(Gromov) hyperbolic* if there exists a $\delta > 0$ such that

$$\langle y, z \rangle_x \geqslant \min\{\langle y, w \rangle_x, \langle z, w \rangle_x\} - \delta$$

for all $x, y, z, w \in G$, where $\langle ., . \rangle$ denotes the *Gromov product* defined by

$$\langle y, z \rangle_x = \frac{1}{2}\left(d(y, x) + d(x, z) - d(y, z)\right).$$

Given a metric space (X, d), we call an isometry $\gamma : [0, L] \to X$ a *geodesic* from $x = \gamma(0)$ to $y = \gamma(L)$. We call a metric space (X, d) *geodesic* if any two points can be connected by such a geodesic. A *geodesic triangle* in (X, d) is a triple $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_i : [0, L_i] \to X$ geodesics and $x = \gamma_3(L_3) = \gamma_1(0)$, $y = \gamma_1(L_1) = \gamma_2(0)$, $z = \gamma_2(L_2) = \gamma_3(0)$. We call x, y and z the *vertices* of the geodesic triangle.

For geodesic metric spaces, Gromov proved the following characterizations of hyperbolicity in terms of geodesic triangles in [Gro87]. Proofs can also be found in [BH99, Proposition III.H.1.17 and Proposition III.H.1.22].

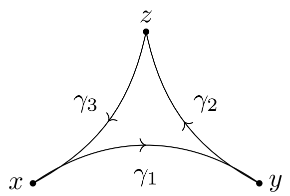


Figure 2.1: A geodesic triangle in a “negatively curved” space

Theorem 2.2.2. *Let (X, d) be a geodesic metric space. Then, the following are equivalent.*

(i) (X, d) *is Gromov hyperbolic.*

(ii) *There exists a $\delta > 0$ such that every geodesic triangle in (X, d) is δ-slim, in the sense that for any geodesic triangle we have that each side is contained in a δ-neighborhood of the union of the other two.*

(iii) *There exists a $\delta > 0$ such that every geodesic triangle in (X, d) is δ-thin, in the sense that for every geodesic triangle with vertices x, y, z and all points p, q on the triangle with $d(x, p) = d(x, q) \leqslant \langle y, z \rangle_x$, we have $d(p, q) \leqslant \delta$.*

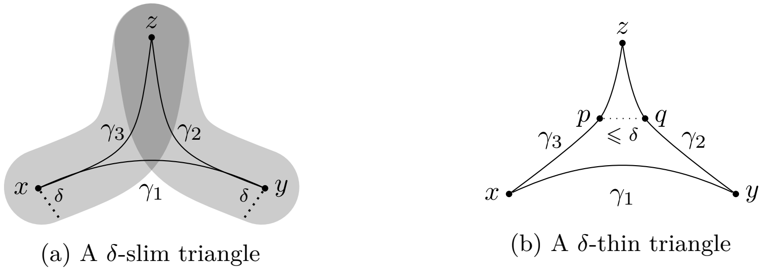


Figure 2.2: Slim and thin geodesic triangles

Hyperbolicity for geodesic metric spaces is a quasi-isometry invariant (see for instance [BH99, Theorem III.H.1.9]).

Recall that the *Cayley graph* of a compactly generated locally compact group G with compact, symmetric generating set $K \subseteq G$ is an undirected graph $\mathcal{G}$ with vertices $V = G$ and with an edge between g and h if $h^{-1}g \in K$. The graph distance defines a metric d on G satisfying

$$d(g,h) = \begin{cases} 0 & \text{if } g = h, \\ \min\{n \in \mathbb{N} \mid h^{-1}g \in K^n\} & \text{if } g \neq h, \end{cases}$$

for all $g, h \in G$. If K' is another compact, symmetric generating set for G, then the Cayley graph $\mathcal{G}'$ with respect to K' is quasi-isometric to $\mathcal{G}$, where the quasi-isometry is given by the identity map on the vertices. One should note that when G is nondiscrete, this Cayley graph is not locally finite and often the action of G on its Cayley graph is not continuous.

Definition 2.2.3. A compactly generated locally compact group is called *hyperbolic* if one (and hence all) of its Cayley graphs is Gromov hyperbolic.

By [CCMT15, Corollary 2.6], a locally compact group G is hyperbolic if and only if G admits a proper, continuous, cocompact, isometric action on a proper geodesic hyperbolic metric space.

Ozawa proved the following in [Oza07].

Proposition 2.2.4. *Let G be a locally compact group admitting a continuous, proper action on a hyperbolic graph with uniformly bounded degree. Then, G is weakly amenable.*

Together with Theorem 2.1.8 along with some structural results on hyperbolic locally compact groups, we have the following.

Corollary 2.2.5. *All locally compact hyperbolic groups are weakly amenable.*

Proof. By [CCMT15, Corollary 2.6], G admits a proper, continuous, cocompact, isometric action on a proper geodesic hyperbolic metric space. By [MMS04, Theorem 21 and Proposition 8], G satisfies at least one of the following three structural properties: G is amenable, or G admits a proper action on a hyperbolic graph with uniformly bounded degree, or G admits closed subgroups $K < G_0 < G$ such that G_0 is of finite index and open in G, K is a compact normal subgroup of G_0 and G_0/K is a real rank one, connected, simple Lie group. Since G_0/K is weakly amenable by Theorem 2.1.8, the results follow from Proposition 2.1.10. □

2.3 C*-algebras

In this section we recall some basic notions about C*-algebras. More details on the basic theory of C*-algebras can for instance be found in [Dav96; Con85].

Recall that a *Banach algebra* is a complex normed algebra A which is complete as a metric space and such that the norm satisfies

$$\|xy\| \leqslant \|x\| \, \|y\| .$$

A *C*-algebra* is a Banach algebra A with a conjugate linear involution $* : A \to A$, called the *adjoint*, that satisfies $(x^*)^* = x$, $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$, $(xy)^* = y^* x^*$ and $\|xx^*\| = \|x\|^2$ for all $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. A C*-algebra A is called *unital* if it contains a neutral element 1 for multiplication.

A $*$-morphism between two $*$-algebras A and B is an algebra morphism $\Phi : A \to B$ satisfying $\Phi(x^*) = \Phi(x)^*$ for all $x \in A$. A bijective $*$-morphism is called a *$*$-isomorphism*. Two C*-algebras are said to be *isomorphic* if there exists a $*$-isomorphism $\Phi : A \to B$. Injective $*$-morphisms $\Phi : A \to B$ between two C*-algebras preserves the norm.

A $*$-morphism $\pi : A \to B(\mathcal{H})$ is called a representation of A. We say that π is *faithful* if it is injective. We say that π is *nondegenerate* if the linear span of $\pi(x)\xi$ for $x \in A$ and $\xi \in \mathcal{H}$ is dense in $\mathcal{H}$.

For any Hilbert space $\mathcal{H}$, the algebra $B(\mathcal{H})$ is a C*-algebra, where the $*$-operation on a $T \in B(\mathcal{H})$ is given by taking the adjoint operator. Moreover, all $\|.\|$-closed $*$-subalgebras of $B(\mathcal{H})$ are C*-algebras. By a theorem of Gelfand and Naimark, every C*-algebra is of this form.

Given a locally compact space X, we denote by $C_0(X)$ the algebra of continuous functions $f : X \to \mathbb{C}$ converging to zero at infinity. The algebra $C_0(X)$ is a C*-algebra for pointwise addition and multiplication, and with involution given by taking the complex conjugate pointwise. The norm is defined as

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

By Gelfand's theorem, every commutative C*-algebra A is isomorphic to $C_0(X)$ for some locally compact space X. The space X is uniquely defined by A up to homeomorphism and is called the *spectrum* of the algebra A. This spectrum is compact if and only if A is unital. Moreover, the category of commutative unital C*-algebras with $*$-morphism can be viewed as dual to the category of compact topological spaces with continuous functions, in the sense that every $*$-morphism $C(X) \to C(Y)$ between commutative, unital C*-algebras induces a continuous map $Y \to X$ and vice versa.

Another important example is the following.

Example 2.3.1. Let A be a C*-algebra and X a locally compact space. The algebra $C_0(X; A)$ of $\|.\|$-continuous, A-valued functions $f : X \to A$ satisfying

$\lim_{x\to\infty}\|f(x)\| = 0$ is a C*-algebra for pointwise addition, multiplication and involution. The norm is given by

$$\|f\|_\infty = \sup_{x\in X}\|f(x)\|$$

for all $f \in C_0(X;A)$.

The *spectrum* of an element x in a C*-algebra A is defined as

$$\sigma(x) = \{\lambda \in \mathbb{C} \mid (\lambda 1 - x) \text{ is not invertible in } A\}.$$

It is always a nonempty compact subset of $\mathbb{C}$. We call a self-adjoint element x *positive* and write $x \geqslant 0$ if $\sigma(x) \subseteq [0,\infty]$. One can prove that $x \in A$ is positive if and only if there exists a $y \in A$ such that $x = y^*y$. The set of positive elements of A is denoted by A^+. We write $x \geqslant y$ if $x - y \geqslant 0$.

Many interesting examples of C*-algebras do not contain a unit (see for instance Sections 2.3.2 and 2.3.3). However, every C*-algebra A contains an *approximate identity*, i.e. a net $(x_i)_i$ of positive elements in A with $\|x_i\| \leqslant 1$ and such that $\|x_i y - y\| \to 0$ for all $y \in A$ (see for instance [Bla06, Proposition II.4.1.3]).

A linear map $\Phi : A \to B$ between two C*-algebras A and B is called *positive* if it maps positive elements of A to positive elements of B. Every positive linear map Φ is bounded and satisfies $\Phi(x^*) = \Phi(x)^*$. Moreover, if A and B are unital, then Φ is positive if and only if $\|\Phi(1)\| = \|\Phi\|$ (see for instance [Bla06, p. II.6.9.4]). A positive linear *functional* $\omega : A \to \mathbb{C}$ with $\omega(1) = 1$ is called a *state*. A positive linear functional is said to be *faithful* if $\omega(x^*x) = 0$ implies $x = 0$.

A linear map $\Phi : A \to B$ is called *completely positive (c.p.)* if the induced maps $\Phi^{(n)} : M_n(A) \to M_n(B)$ defined by $\left(\Phi^{(n)}(A)\right)_{i,j} = \Phi^{(n)}(A_{i,j})$ are positive for all $n \in \mathbb{N}\setminus\{0\}$. If A and B are unital and $\Phi(1) = 1$, we say that Φ is *unital and completely positive (u.c.p.)*. Every positive linear map $\Phi : A \to B$ where either A or B is commutative is automatically completely positive (see [Sti55, Theorem 4] for when B is commutative and [Pau03, Theorem 3.9] for when A is commutative). In particular, every positive linear functional is completely positive [Sti55, Theorem 3].

A linear map $\Phi : A \to B$ is called *completely bounded (c.b.)* if

$$\|\Phi\|_{\mathrm{cb}} = \sup_n \left\|\Phi^{(n)}\right\| < \infty.$$

Note that since $\left\|\Phi^{(n)}\right\| = \|\Phi(1)\|$ when Φ is completely positive, all completely positive maps are completely bounded. A completely bounded map Φ with

$\|\Phi\|_{\mathrm{cb}} \leqslant 1$ is called a *complete contraction (c.c.)*. If Φ is also positive, we call Φ *contractive and completely positive (c.c.p.)* A good reference for completely positive and completely bounded maps is [Pau03].

We also mention the following theorem due to Stinespring for completely positive maps (see [Sti55, Theorem 1]) and Paulsen for completely bounded maps (see [Pau84, Theorem 2.7]).

Theorem 2.3.2 (Stinespring Dilation Theorem)**.** *Let A be a C*-algebra and $\Phi : A \to B(\mathcal{H})$ a completely bounded map. Then, there exists a Hilbert space $\mathcal{K}$, a $*$-morphism $\pi : A \to B(\mathcal{K})$ and bounded operators $V, W : \mathcal{H} \to \mathcal{K}$ with $\|\Phi\|_{cb} = \|V\| \, \|W\|$ such that*

$$\Phi(x) = W^* \pi(x) V$$

for all $x \in A$.

2.3.1 Unitization

Every nonunital C*-algebra A can be extended in a number of ways to a unital C*-algebra B that contains A as an *essential ideal*, meaning that if a $y \in B$ satisfies $xy = 0$ for all $x \in A$, then $y = 0$. These extensions are called *unitizations* of the C*-algebra A.

Let A be a nonunital C*-algebra. The easiest way to extend A to a unital C*-algebra is to consider the algebra $A^\dagger = A \oplus \mathbb{C}$ with the addition and the multiplication and involution given by

$$(x, \lambda)(y, \mu) = (xy, \mu x + \lambda y + \lambda\mu) \qquad \text{and} \qquad (x, \lambda)^* = (x^*, \overline{\lambda})$$

for $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. The $*$-algebra $A^\dagger$ is a C*-algebra with respect to the following norm

$$\|(x, \lambda)\| = \sup_{\substack{y \in A \\ \|y\| \leqslant 1}} \|xy + \lambda y\|$$

for $x \in A$ and $\lambda \in \mathbb{C}$ (see [Con85, Proposition VII.1.9]). Clearly, $A^\dagger$ contains A as a closed ideal and $A^\dagger / A \cong \mathbb{C}$ and this ideal is essential. Hence, $A^\dagger$ is indeed a unitization of A. It is the smallest unitization of A in the following sense. A proof of this result can be found in [Con85, Proposition VII.1.9].

Proposition 2.3.3. *Let A be a nonunital C*-algebra. If B is a unital C*-algebra A, then the natural embedding $i : A \hookrightarrow B$ extends uniquely to a unital embedding $\tilde{i} : A^\dagger \hookrightarrow B$ given by*

$$\tilde{i}(a, \lambda) = i(a) + \lambda 1_B$$

for $a \in A$ and $\lambda \in \mathbb{C}$.

When A is faithfully represented on a Hilbert space $\mathcal{H}$, then

$$A^\dagger \cong \{x + \lambda 1_{\mathcal{H}}\}_{x\in A, \lambda\in\mathbb{C}}.$$

Another unitization of a nonunital C*-algebra A is the so-called *multiplier algebra* $M(A)$. Consider the A-valued inner product $\langle\cdot,\cdot\rangle$ on A defined by

$$\langle x, y\rangle = x^* y$$

for $x, y \in A$. Note that $\|x\|^2 = \langle x, x\rangle$ for all $x \in A$. This inner product turns A into a Hilbert C*-module (see [Lan95] for more details). An operator $T : A \to A$ is called *adjointable* if there exists an operator $S : A \to A$ such that

$$\langle Tx, y\rangle = \langle x, Sy\rangle$$

for all $x, y \in A$. The operator S is unique and is called the adjoint T^* of T. Note that any adjointable operator automatically commutes with the action of right multiplication on A, i.e. $(Tx)y = T(xy)$ for any $x, y \in A$ and any adjointable operator $T : A \to A$. Moreover, the set of adjointable operators $M(A)$ is closed under addition, composition and taking the adjoint. Equipping $M(A)$ with the operator norm, one even proves that it is a C*-algebra.

Definition 2.3.4. Let A be a nonunital C*-algebra. The C*-algebra $M(A)$ is adjointable operators $T : A \to A$ is called the *multiplier algebra* of A.

Again, $M(A)$ contains A as a closed essential ideal. Indeed, for every $x \in A$, the operator $L_x : A \to A$ defined by $L_x y = xy$ is adjointable (with adjoint L_{x^*}). The multiplier algebra $M(A)$ is the largest unitization in the following sense.

Proposition 2.3.5. *Let A be a nonunital C*-algebra and B a unitization of A. Then, the natural embedding $A \hookrightarrow M(A)$ extends uniquely to a unital embedding $B \hookrightarrow M(A)$.*

Example 2.3.6. Let X be a locally compact space. Denote by $C_b(X)$ the algebra of bounded continuous functions $f : X \to \mathbb{C}$. Denote $A = C_0(X)$. For every $f \in C_b(X)$, the operator $L_f : A \to A$ defined by $L_f(h) = fh$ is an adjointable operator. Looking at the images of an approximate unit under an adjointable operator $T : A \to A$, one even proves that any such operator is of that form and hence $M\big(C_0(X)\big) \cong C_b(X)$.

When A is faithfully and nondegenerately represented on a Hilbert space $\mathcal{H}$, then one can prove that

$$M(A) \cong \{T \in B(\mathcal{H}) \mid xT \in A \text{ and } Tx \in A \text{ for all } x \in A\}. \tag{2.3.1}$$

Since the algebra $K(\mathcal{H})$ of compact operators on $\mathcal{H}$ is an ideal in $B(\mathcal{H})$, this implies that the multiplier algebra of $K(\mathcal{H})$ is $B(\mathcal{H})$

Apart from the norm topology, one also considers the following topology on $M(A)$.

Definition 2.3.7. Let A be a C*-algebra. The *strict topology* on $M(A)$ is the weakest topology for which the maps $T \mapsto \|Tx\|$ and $T \mapsto \|T^*x\|$ for $x \in A$ are continuous.

Using an approximate unit, one checks that A is strictly dense in $M(A)$. Moreover, one can even prove that the unit ball of A is strictly dense in the unit ball of $M(A)$ (see [Lan95, Proposition 1.4]).

Example 2.3.8. In the same way as Example 2.3.6, one proves that the multiplier algebra of $C_0(X; A)$ from Example 2.3.1 is the algebra $C_b^{\mathrm{str}}\big(X; M(A)\big)$ of bounded, strictly continuous, $M(A)$-valued functions $f : X \to M(A)$. See [APT73, Corollary 3.4] for more details.

Unitizations of commutative C*-algebras

By Gelfrand's theorem, every nonunital commutative C*-algebra A is isomorphic to $C_0(X)$, where X a locally compact, noncompact space, called the spectrum of A. The unitizations of the algebra A correspond to the compactifications of the spectrum of A. A compactification is a topological space X is defined as follows.

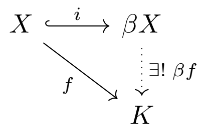
Definition 2.3.9. Let X be a locally compact space. We call a compact space Y a *compactification* of X if Y contains X as a dense open subset.

The correspondence between unitizations of $A \cong C_0(X)$ and compactifications of X now goes as follows. Let B be a unitization of A. The spectrum of B is a compact space Y. Since A is an ideal of B, the space Y contains X as an open subset. Moreover, since A is an *essential* ideal, the subset X is dense in Y. Hence, Y is indeed a compactification of Y. Conversely, if Y is a compactification of X, then it is easy to see that $A \cong C_0(X)$ is an essential ideal in $C(Y)$.

The easiest example of a compactification, is the so-called *one-point compactification* $X^\dagger$ of X, consisting of the space X together with a single point x_∞ (see for instance [Mun00, Theorem 29.1]). From the universal property Proposition 2.3.3, it follows immediately that $C(X^\dagger) \cong C_0(X)^\dagger$.

Recall that $M(A) \cong C_b(X)$. The spectrum of this algebra is denoted by βX and is called the *Stone-Čech compactification*. It is characterized by the following universal property. A proof can for instance be found in [Mun00, Lemma 38.1].

Theorem 2.3.10. *Let X be a locally compact space. Then, the Stone-Čech compactification βX along with the inclusion $i : X \to \beta X$ satisfies the following universal property: for every compact space K and any continuous map $f : X \to K$, there exists a unique continuous map $\beta f : \beta X \to K$ making the following diagram commute.*



Note that by Proposition 2.3.5, the unitizations of $A = C_0(X)$ are the unital C*-algebras B such that $C_0(X) \subseteq B \subseteq C_b(X)$. Hence, by the previous, the compactifications X correspond to the spectra of these algebras.

Suppose that G is a locally compact group with a continuous action $G \curvearrowright X$. This action induces an action $G \curvearrowright^{\alpha} A = C_0(X)$ by $*$-automorphisms defined by

$$(\alpha_g f)(x) = f(g^{-1}x)$$

for $f \in A$ and $x \in X$. Moreover, this action is point-norm continuous in the sense that

$$\lim_{g \to h} \|\alpha_g(f) - \alpha_h(f)\|_\infty \to 0$$

for every $f \in A$. Such a triple (A, G, α) is called a *C*-dynamical system*. By Gelfand duality, every C*-dynamical system (A, G, α) with a continuous C*-algebra A, gives rise to a continuous action $G \curvearrowright X$ on the spectrum X of A.

In general, a continuous action $G \curvearrowright X$ does not extend to a continuous action on a compactification $G \curvearrowright Y$. However, on some compactifications it does.

Definition 2.3.11. Let G be a locally compact group, X a locally compact space and $G \curvearrowright X$ a continuous action. A compactification Y of X is called *equivariant* for the action $G \curvearrowright X$ if this action extends to a continuous action $G \curvearrowright Y$.

Clearly, the one point compactification $X^\dagger$ is equivariant for any action. The largest equivariant compactification of X is the spectrum $\beta^G X$ of the algebra

$$C_b^G(X) = \left\{ f \in C_b(X) \;\middle|\; \|\sigma_g f - f\|_\infty \to 0 \text{ if } g \to e \right\}.$$

It is easy to see that $\beta^G X$ is indeed G-equivariant and that the inclusion $i : X \to \beta^G X$ given by Gelfand duality is G-equivariant. Moreover, it satisfies the following universal property.

Theorem 2.3.12. *Let G be a locally compact group, X a locally compact space and $G \curvearrowright X$ a continuous action. The space $\beta^G X$, together with the natural G-equivariant inclusion $i : X \to \beta^G X$, satisfies the following universal property: for every compact space K with continuous action $G \curvearrowright K$ and any continuous, G-equivariant map $f : X \to K$, there exists a unique continuous, G-equivariant map $\beta^G f : \beta^G X \to K$ making the following diagram commute*

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \beta^G X \\ & \searrow^{f} & \vdots\ \exists!\ \beta^G f \\ & & K \end{array}$$

We call the space $\beta^G X$ the *equivariant Stone-Čech compactification* for $G \curvearrowright X$.

There is a one-to-one correspondence between the equivariant compactifications Y of X and the C*-dynamical systems (B, G, α), where $A \subseteq B \subseteq C_b^G(X)$ and the inclusions are G-equivariant. Indeed, every equivariant compactification corresponds to an algebra B with $A \subseteq B \subseteq C_b(X)$, and since (B, G, α) forms a C*-dynamical system, we must have that $\lim_{g\to h} \|\alpha_g(f) - \alpha_h(f)\|_\infty \to 0$ for $f \in B$ and hence $B \subseteq C_b^G(X)$. Conversely, if (B, G, α) is a C*-dynamical system, then the spectrum Y of B is equipped with a continuous action $G \curvearrowright Y$. Since the inclusion $A \subseteq B$ is G-equivariant, the action $G \curvearrowright Y$ is indeed an extension of $G \curvearrowright X$.

When $G \curvearrowright G$ by left translation, the equivariant Stone-Čech compactification is the spectrum of the algebra $C_b^{lu}(G)$ of bounded left uniformly continuous functions $f : G \to \mathbb{C}$. We will denote its spectrum by $\beta^{lu} G$. Similarly, we denote by $\beta^{ru} G$ and $\beta^u G$ the equivariant Stone-Čech compactification for the actions $G \curvearrowright G$ by right translation and $G \times G \curvearrowright G$ by left and right translation respectively. The space $\beta^{ru} G$ (resp. $\beta^u G$) is the spectrum of the algebra $C_b^{ru}(G)$ (resp. $C_b^u(G)$) of right uniformly (resp. left and right uniformly) continuous functions $f : G \to \mathbb{C}$.

2.3.2 Group C*-algebras

In this section, we discuss the construction of the group C*-algebra. Proofs of and references for the results mentioned here can be found in [Bla06, Section II.10.2] and [Wil07], or in [BO08, Section 2.5] for the particular case of discrete groups.

Let G be a locally compact group. Consider the space $C_c(G)$ of continuous functions $f : G \to \mathbb{C}$ with compact support. Equipping $C_c(G)$ with the *convolution product* defined by

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)\,\mathrm{d}s \tag{2.3.2}$$

and involution defined by

$$(f^*)(t) = \delta_G(t^{-1})\overline{f(t^{-1})} \tag{2.3.3}$$

for $f, g \in C_c(G)$ and $t \in G$, we turn $C_c(G)$ into a $*$-algebra.

Given a s.o. continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, we define for every $f \in C_c(G)$ the operator $\pi(f) \in B(\mathcal{H})$ by

$$\pi(f) = \int_G f(t)\pi_t\,\mathrm{d}t,$$

where the integral denotes the Bochner integral (see for instance [DS58]). This defines a nondegenerate $*$-morphism $\pi : C_c(G) \to B(\mathcal{H})$ satisfying $\|\pi(f)\|_1 \leqslant \|f\|_1$.

In particular, the *left regular representation* $\lambda : G \to \mathcal{U}(L^2(G))$ defined by

$$(\lambda_g\xi)(t) = \xi(g^{-1}t)$$

for $g, t \in G$ and $\xi \in L^2(G)$ induces a $*$-morphism $\lambda : C_c(G) \to B(\mathcal{H})$ satisfying

$$(\lambda(f)\xi)(t) = \int_G f(s)\xi(s^{-1}t)\,\mathrm{d}s \tag{2.3.4}$$

for all $f \in C_c(G)$, $\xi \in L^2(G)$ a.e. $t \in G$.

The reduced and the full group C*-algebras are now defined as follows.

Definition 2.3.13. Let G be a locally compact group.

(a) The *(full) group C*-algebra* $\mathrm{C}^*(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\| = \sup_\pi \|\pi(f)\|$$

for $f \in C_c(G)$, where the supremum runs over all s.o. continuous unitary representations π of G.

(b) The *reduced group C*-algebra* $\mathrm{C}^*_r(G)$ is the completion of $C_c(G)$ with respect to the norm $\|f\|_r = \|\lambda(f)\|$, where λ is the left regular representation as before. In other words,

$$\mathrm{C}^*_r(G) \cong \overline{\{\lambda(f)\}_{f \in C_c(G)}}$$

It is clear from the definition of the full group C*-algebra that every s.o. continuous unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ induces a $*$-morphism $\tilde{\pi}(f) : \mathrm{C}^*(G) \to B(\mathcal{H})$ satisfying $\tilde{\pi}(f) = \pi(f)$ for every $f \in C_c(G)$. In particular, there is a surjective $*$-morphism $\mathrm{C}^*(G) \to \mathrm{C}^*_r(G)$. This $*$-morphism is an isomorphism if and only if G is amenable (see [Pat88, Theorem 4.21]). Note that neither the full nor the reduced C*-algebra is unital, unless G is discrete.

The reduced C*-algebra $\mathrm{C}^*_r(G)$ is canonically isomorphic to the C*-algebra induced by the right regular representation, i.e. the completion of $C_c(G)$ with respect to the norm $\|f\| = \|\rho(f)\|$. It follows that $\mathrm{C}^*_r(G)$ has two natural representations on $L^2(G)$: the representation $\lambda : G \to B\big(L^2(G)\big)$ induced by the left regular representation and the representation $\rho : \mathrm{C}^*_r(G) \to B\big(L^2(G)\big)$ induced by the right regular representation. Note that that latter one satisfies

$$(\rho(f)\xi)(t) = \int_G f(s)(\rho_s\xi)(t)\,\mathrm{d}s = \int_G f(s)\delta_G(s)^{1/2}\xi(ts)\,\mathrm{d}s \tag{2.3.5}$$

for all $f \in \mathrm{C}^*_r(G)$, $\xi \in L^2(G)$ and a.e. $t \in G$.

The multiplier algebra of both $\mathrm{C}^*(G)$ and $\mathrm{C}^*_r(G)$ contains a group of unitaries $\{u_g\}_{g\in G}$ satisfying

$$(u_g f)(t) = f(g^{-1}t) \qquad \text{and} \qquad (fu_g)(t) = f(tg^{-1})\delta_G(g)^{-1}$$

for every $g, t \in G$ and every $f \in C_c(G)$. In the representation $\lambda : \mathrm{C}^*_r(G) \to B\big(L^2(G)\big)$ this group of unitaries is given by the operators $\{\lambda_g\}$ from the left left regular representation $\lambda : G \to \mathcal{U}\big(L^2(G)\big)$.

2.3.3 Crossed products

Another typical class of examples of C*-algebras are the crossed products. The crossed product construction is a generalization of the group C*-algebra construction above. Again we refer to [Bla06, Section II.10.3], [Wil07] and [BO08, Section 4.1] for more details. Recall that a *C*-dynamical system* or *covariant system* is a triple (A, G, α) consisting of a C*-algebra A, a locally compact group G and an action $G \curvearrowright^{\alpha} A$ by automorphisms that is point-norm continuous, i.e. such that

$$\lim_{g\to h} \|\alpha_g(x) - \alpha_h(x)\| = 0$$

for every $x \in A$.

Consider the algebra $C_c(G, A)$ of $\|.\|$-continuous functions $G \to A$ with compact support. We equip $C_c(G, A)$ with the obvious pointwise addition along with the twisted convolution product and involution defined by

$$(f * g)(t) = \int_G f(s)\alpha_s\big(g(s^{-1}t)\big)\,\mathrm{d}s \qquad \text{and} \qquad f^*(t) = \delta_G(t^{-1})\alpha_t\big(f(t^{-1})\big)^*$$

for all $f, g \in C_c(G, A)$ and $t \in G$. Again, this turns $C_c(G, A)$ into a $*$-subalgebra.

A *covariant representation* of a C*-dynamical system (A, G, α) on $\mathcal{H}$ is a pair (π, ρ) consisting of a $*$-representation $\pi : A \to B(\mathcal{H})$ and an s.o. continuous unitary representation $\rho : G \to \mathcal{U}(\mathcal{H})$ on *the same* Hilbert space $\mathcal{H}$ such that the relation

$$\rho_g \pi(f)\rho_g^* = \pi\big(\alpha_g(f)\big)$$

holds. We say that (π, ρ) is nondegenerate if π is nondegenerate. Given any nondegenerate covariant representation (π, ρ) of (A, G, α) on $\mathcal{H}$, the associated $*$-morphism is defined as $(\pi \rtimes \rho) : C_c(G, A) \to B(\mathcal{H})$ by

$$(\pi \rtimes \rho)(f) = \int_G \pi\big(f(t)\big)\rho_t\,\mathrm{d}t$$

for $f \in C_c(G, A)$. As in the case of group C*-algebras, $\pi \rtimes \rho$ is a well-defined $*$-morphism satisfying

$$\|(\pi \rtimes \rho)(f)\| \leqslant \|f\|_1 = \int_G \|f(t)\|\,\mathrm{d}t.$$

The full crossed product is now defined as follows.

Definition 2.3.14. Let (G, A, α) be a C*-dynamical system. The *(full) crossed product* $A \rtimes_\alpha G$ of (A, G, α) is the completion of $C_c(G, A)$ with respect to the norm

$$\|f\| = \sup \|(\pi \rtimes \rho)(f)\|$$

where the supremum runs over all is a nondegenerate covariant representations (π, ρ) of (A, G, σ).

Again, every nondegenerate covariant representation (π, ρ) of (A, G, σ) on $\mathcal{H}$ induces a (nondegenerate) representation $\pi \rtimes \rho : A \rtimes_\sigma G \to B(\mathcal{H})$.

Fix a faithful $*$-representation of A on a Hilbert space $\mathcal{H}$. Denote $\mathcal{K} = \mathcal{H} \otimes L^2(G)$. There is a natural covariant representation (π_σ, u) of (A, G, σ) on $\mathcal{K}$ given by

$$\big(\pi_\alpha(x)\xi\big)(t) = \sigma_t^{-1}(x)\big(\xi(t)\big) \qquad \text{and} \qquad (u_s\xi)(t) = \xi(s^{-1}t)$$

for $s, t \in G$, $x \in A$ and $\xi \in \mathcal{K}$. The reduced crossed product is now defined as follows.

Definition 2.3.15. Let (A, G, α) be a C*-dynamical system. The *reduced crossed product* $A \rtimes_{\alpha,r} G$ of (A, G, α) is the completion of $C_c(G, A)$ with respect to the norm

$$\|f\|_r = \|(\pi_\alpha \rtimes u)(f)\|,$$

where π_α and u are as above. In other words,

$$A \rtimes_{\alpha,r} G \cong \overline{\{(\pi_\alpha \rtimes u)(f)\}_{f\in C_c(G,A)}}.$$

This definition does not depend on the choice of the faithful representation π above (see for instance [Wil07, Definition 7.7]). As before, there exists a natural quotient map $A \rtimes G \to A \rtimes_r G$ which is injective in some specific cases (see Theorem 2.3.30). Neither the full nor the reduced crossed product is unital, unless G is discrete and A is unital.

As in the case of the group C*-algebras, the multiplier algebra of both the full and the reduced crossed product contains a group of unitaries $\{u_g\}_{g\in G}$ satisfying

$$(u_g f)(t) = \sigma_g\big(f(g^{-1}t)\big) \qquad \text{and} \qquad (fu_g)(t) = f(tg^{-1})\delta_G(g)^{-1}$$

for all $g, t \in G$ and all $f \in C_c(G, A)$. The multiplier algebras also contain a copy of A satisfying

$$(xf)(t) = xf(t) \qquad \text{and} \qquad (fx)(t) = f(t)\alpha_t(x)$$

for every $x \in A$, $f \in C_c(G, A)$ and $t \in G$.

Transformation group C*-algebras

A special case of the previous crossed product construction is the following. Suppose that X is a locally compact space and $G \curvearrowright X$ is a continuous action. Denoting $A = C_0(X)$, the given action induces an action $G \curvearrowright^\alpha A$ defined by

$$(\alpha_g f)(x) = f(g^{-1}x)$$

for $f \in A$ that turns (A, G, α) into a C*-dynamical system. The full and reduced crossed product of this C*-dynamical system is called the *full and reduced transformation group C*-algebra* respectively.

In this setting, we can view the algebra $C_c(X \times G)$ of continuous, compactly supported functions $X \times G \to \mathbb{C}$ as a subalgebra of $C_c(G, A)$ by identifying a function $f \in C_c(X \times G)$ with the function $g \mapsto f(\cdot, g)$. Under this identification, the multiplication and involution on $C_c(X \times G)$ are given by

$$(f * g)(x, t) = \int_G f(x, s)g(s^{-1}x, s^{-1}t)$$

and

$$(f^*)(t) = \delta_G(t^{-1})\overline{f(t^{-1}x, t^{-1})}$$

for $f, g \in C_c(X \times G)$, $x \in X$ and $t \in G$. Moreover, note that $C_c(X \times G)$ is dense inside $C_c(G, A)$ for the $\|.\|_1$ given by

$$\|f\|_1 = \int_G \|f(t)\| \, \mathrm{d}t = \int_G \sup_{x \in X} |f(t,x)| \, \mathrm{d}t.$$

Hence, $C_c(X \times G)$ can be viewed as a dense $*$-subalgebra of both the full and the reduced crossed product of (A, G, σ).

If we equip X with a Radon measure assigning a strictly positive measure to every open subset, we can faithfully represent A on $\mathcal{H} = L^2(X)$ as multiplication operators. The representation $\pi_\sigma \rtimes u$ used in the definition of the reduced crossed product $A \rtimes_r G$ is then given by

$$\big((\pi_\sigma \rtimes u)(f)\xi\big)(x,t) = \int_G f(tx, s)\xi(x, s^{-1}t) \, \mathrm{d}s \tag{2.3.6}$$

for every $f \in C_c(X \times G)$, every $\xi \in \mathcal{K} = L^2(X) \otimes L^2(G)$, a.e. $t \in G$ and a.e. $x \in X$. Using the unitary $U : \mathcal{K} \to \mathcal{K}$ defined by

$$(U\xi)(x,t) = \xi(t^{-1}x, t)$$

for $\xi \in \mathcal{K}$, $x \in X$ and $t \in G$, one checks that the covariant representation (π_σ, u) is unitarily equivalent with the covariant representation (π'_σ, u') given by

$$\pi'_\sigma(f) = U\pi_\sigma(f)U^* = f \otimes 1 \qquad \text{and} \qquad u'_g = U u_g U^* = \sigma_g \otimes \lambda_g.$$

The associated $*$-morphism $\pi'_\sigma \rtimes u'$ is then given by

$$\begin{aligned}\big((\pi'_\sigma \rtimes u')(f)\xi\big)(x,t) &= \big(U(\pi_\sigma \rtimes u)(f)U^*\xi\big)(x,t) \\ &= \int_G f(x,s)\xi(s^{-1}x, s^{-1}t) \, \mathrm{d}s \end{aligned} \tag{2.3.7}$$

for all $f \in C_c(X \times G)$, all $\xi \in \mathcal{K}$, a.e. $x \in X$ and a.e. $t \in G$. Hence,

$$A \rtimes_r G \cong \overline{\{(\pi'_\sigma \rtimes u')\}_{f \in C_c(G \times X)}}.$$

2.3.4 Tensor products

Let A and B be two C*-algebras. The algebraic tensor product $A \otimes_{\mathrm{alg}} B$ is a $*$-algebra when equipped with the obvious addition, multiplication and involution.

To turn this $*$-algebra into a C*-algebra, one has to complete it in some way. There are a number of ways to do this. Below, we discuss the two most common ways. More information and proofs can be found in [BO08, Chapter 3], [Bla06, p. II.9.2] and [Tak79, Section IV.4].

Given two faithful $*$-representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{K})$, there exists a unique $*$-representation $\pi_A \otimes \pi_B : A \otimes_{\text{alg}} B \to B(\mathcal{H} \otimes \mathcal{K})$ satisfying

$$(\pi_A \otimes \pi_B)(x \otimes y) = \pi_A(x) \otimes \pi_B(y)$$

for all $x \in A$ and $y \in B$. The closure of the image of $\pi_A \otimes \pi_B$ inside $B(\mathcal{H} \otimes \mathcal{K})$ forms a C*-algebra. One can prove that for all choices of faithful $*$-representations π_A and π_B, the resulting algebra is isomorphic. This yields the following definition.

Definition 2.3.16. Let A and B be two C*-algebras. Let $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{K})$ be two faithful $*$-representations. The *minimal* or *spatial tensor product* $A \otimes_{\min} B$ is the completion of $A \otimes_{\text{alg}} B$ with respect to the norm

$$\left\| \sum_i x_i \otimes y_i \right\|_{\min} = \left\| (\pi_A \otimes \pi_B)\Big(\sum_i x_i \otimes y_i \Big) \right\| = \left\| \sum_i \pi_A(x_i) \otimes \pi_B(y_i) \right\|,$$

where $\sum_i x_i \otimes y_i \in A \otimes_{\text{alg}} B$. In other words,

$$A \otimes_{\min} B \cong \overline{\operatorname{span}\{\pi_A(x) \otimes \pi_B(y)\}_{x \in A, y \in B}}.$$

As the notation suggests, one can prove that $\|.\|_{\min}$ is indeed the smallest C*-norm that can be put on $A \otimes_{\text{alg}} B$.

Given two $*$-representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ with commuting ranges on the same Hilbert space $\mathcal{H}$, there exists a unique induced $*$-representation $\pi : A \otimes_{\text{alg}} B \to B(\mathcal{H})$ satisfying

$$\pi(x \otimes y) = \pi_A(x)\pi_B(y)$$

for all $x \in A$ and $y \in B$. Conversely, every $*$-representation of $A \otimes_{\text{alg}} B$ arises in this way: given a $*$-representation $\pi : A \otimes_{\text{alg}} B \to B(\mathcal{H})$ there exist two $*$-representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_B : B \to B(\mathcal{H})$ with commuting range such that $\pi(x \otimes y) = \pi_A(x)\pi_B(y)$ for all $x \in A$ and $y \in B$. This observation lead to the following definition.

Definition 2.3.17. Let A and B be two C*-algebras. The *maximal* or *universal tensor product* $A \otimes_{\max} B$ is the completion of $A \otimes_{\text{alg}} B$ with respect to the norm

$$\|x\|_{\max} = \sup \left\{ \|\pi(x)\| \mid \pi : A \otimes_{\text{alg}} B \to B(\mathcal{H}) \text{ is a } *\text{-representation} \right\},$$

where $x \in A \otimes_{\text{alg}} B$.

Clearly, $\|.\|_{\max}$ is indeed the maximal C*-norm that one can put on $A \otimes_{\text{alg}} B$. Every $*$-morphism $\pi : A \otimes_{\text{alg}} B \to C$ to a C*-algebra C extends to a $*$-morphism $A \otimes_{\max} B \to C$. In particular, any pair of $*$-morphisms $\pi_A : A \to C$ and $\pi_B : B \to C$ with commuting ranges induces a unique $*$-morphism $\pi : A \otimes_{\max} B \to C$ satisfying $\pi(x \otimes y) = \pi_A(x)\pi_B(y)$ for any $x \in A$ and $y \in B$.

The following result allows the construction of maps on tensor products from maps on both components whenever these maps on the components are 'nice' enough.

Theorem 2.3.18. *Let A, B, C and D be C*-algebras.*

(a) Given c.b. maps $\Phi : A \to C$ and $\Psi : B \to D$, there exists a unique c.b. map

$$\Phi \otimes \Psi : A \otimes_{min} B \to C \otimes_{min} D$$

satisfying $(\Phi \otimes \Psi)(x \otimes y) = \Phi(x) \otimes \Psi(y)$. Moreover, $\|\Phi \otimes \Psi\|_{cb} = \|\Phi\|_{cb} \|\Psi\|_{cb}$.

(b) Given c.p. maps $\Phi : A \to C$ and $\Psi : B \to D$, there exists a unique map c.p. map

$$\Phi \otimes \Psi : A \otimes_{max} B \to C \otimes_{max} D$$

satisfying $(\Phi \otimes \Psi)(x \otimes y) = \Phi(x) \otimes \Psi(y)$. Moreover, $\|\Phi \otimes \Psi\| = \|\Phi\| \|\Psi\|$.

This result is proven by using the Stinespring Dilation Theorem (see for instance [BO08, Theorem 3.5.3 and Remark 3.5.4]). Note that the corresponding result for the maximal tensor product and c.b. maps is *not* true (see [Hur83]).

We use the following example a number of times during the thesis.

Example 2.3.19. One can prove that the algebra $C_0(X; A)$ from Example 2.3.1 is isomorphic to $C_0(X) \otimes_{\min} A$ (which is in turn isomorphic to $C_0(X) \otimes_{\max} A$, see Section 2.3.5). The isomorphism is given by mapping an elementary tensor $f \otimes a \in C_0(X) \otimes_{\text{alg}} A$ to the function $x \mapsto f(x)a$. See [Bla06, Theorem II.9.4.4] for more information.

The operation of taking the maximal tensor product preserves short exact sequences.

Proposition 2.3.20. *Let*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

be an exact sequence. For every C-algebra D, the natural sequence*

$$0 \longrightarrow A \otimes_{max} D \xrightarrow{i \otimes \mathrm{id}_D} B \otimes_{max} D \xrightarrow{p \otimes \mathrm{id}_D} C \otimes_{max} D \longrightarrow 0$$

is also exact.

A proof of this proposition can be found in [BO08, Proposition 3.7.1]. The same is *not* true the minimal tensor product in general. However, there is a class of C*-algebras for which it is true, see Theorem 2.3.27.

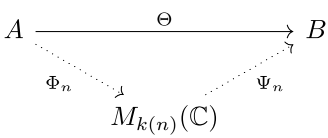
2.3.5 Nuclearity and exactness

Nuclear and exact C*-algebras are two special classes of C*-algebras. We will define these classes in terms of nuclearity of certain maps, and we will see that tensor products of C*-algebras in this class have certain special properties. We will also discuss some examples. Proofs of the results presented here can be found in [BO08, Chapter 2] and [Bla06, Section IV.3].

Definition 2.3.21. Let A and B be two C*-algebras. A c.c.p. map $\Theta : A \to B$ is called *nuclear* if there exist sequences of c.c.p. maps $\Phi_n : A \to M_{k(n)}(\mathbb{C})$ and $\Psi_n : M_{k(n)}(\mathbb{C}) \to B$ such that

$$\|(\Psi_n \circ \Phi_n)(x) - \Theta(x)\| \to 0$$

for all $x \in A$.



It is clear that the composition of c.c.p. maps $\Theta_1 : A \to B$ and $\Theta_2 : B \to C$ is nuclear if either Θ_1 or Θ_2 is nuclear. If both A and B are assumed to be unital, then one can assume that the maps Φ_n and Ψ_n in the definition above are u.c.p.

It is important to note that nuclearity of the map $\Theta : A \to B$ depends on the codomain B: it is possible that Θ is not nuclear, while the same map $\widetilde{\Theta} : A \to B(\mathcal{H}) : x \mapsto \theta(x)$ with extended codomain is nuclear. The issue being that the images of the approximation maps Ψ_n for $\widetilde{\Theta}$ do not need to be in B. We will see that this is exactly the difference between the class of nuclear and the class of exact C*-algebras.

The following result shows that nuclearity of a map behaves well with respect to adding a unit. A proof can be found in [BO08, Proposition 2.2.4].

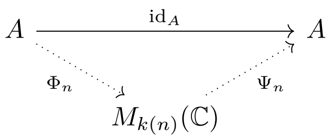
Proposition 2.3.22. *Let A and B be two C^*-algebras and $\Theta : A \to B$ a nuclear map.*

(a) If A is nonunital and B is unital, then the unique linear extension $\Theta^\dagger : A^\dagger \to B$ is also u.c.p. and nuclear.

(b) If both A and B are nonunital, then the unique linear extension $\Theta^\dagger : A^\dagger \to B^\dagger$ is also nuclear.

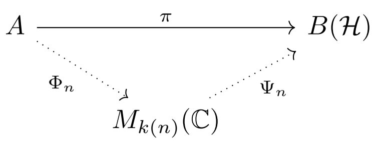
Nuclear and exact C*-algebra as now defined as follows.

Definition 2.3.23. A C*-algebra A is called *nuclear* if the identity map $\mathrm{id}_A : A \to A$ is nuclear.



Other names for nuclear C*-algebras include *amenable* C*-algebras or C*-algebras with *completely positive approximation property (CPAP).*

Definition 2.3.24. A C*-algebra A is called *exact* if there exists a faithful representation $\pi : A \to B(\mathcal{H})$ such that π is nuclear.



Using Arveson's Extension Theorem, it is not difficult to show that in fact every representation $\pi : A \to B(\mathcal{K})$ is nuclear if A is exact. Clearly, all nuclear C*-algebras are exact.

Easy examples of nuclear C*-algebras are abelian C*-algebras and (approximately) finite-dimensional C*-algebras. Subalgebras of exact C*-algebras are exact, but the same is not true for nuclear C*-algebras. Thanks to Proposition 2.3.22, a nonunital C*-algebra A is nuclear (resp. exact) if and only if $A^\dagger$ is nuclear (resp. exact).

The following follows from a deep result due to Kirchberg. Proofs can be found in [BO08, Corollary 9.4.3 and 9.4.4].

Theorem 2.3.25. *Let A be a C*-algebra and $J \subseteq A$ an ideal.*

(a) If A is exact, then so is A/J.

(b) If A is nuclear, then so is A/J.

As mentioned before, the classes of nuclear and exact C*-algebras behave in a special way with respect to tensor products. For nuclear C*-algebras all notions of tensor products coincide.

Theorem 2.3.26. *A C*-algebra A is nuclear if and only if for every other C*-algebra B, we have that $\|.\|_{min}$ and $\|.\|_{max}$ coincide on $A \otimes_{alg} B$. In particular, if A is nuclear then*

$$A \otimes_{max} B \cong A \otimes_{min} B.$$

For exact C*-algebras, the operation of taking the *minimal* tensor product preserves exactness. This was even the original definition of exactness. The proof that this is equivalent with the definition above is due to Kirchberg.

Theorem 2.3.27. *A C*-algebra A is exact if and only if for every short exact sequence*

$$0 \longrightarrow B \xrightarrow{\ i\ } C \xrightarrow{\ p\ } D \longrightarrow 0$$

also the natural sequence

$$0 \longrightarrow B \otimes_{min} A \xrightarrow{i \otimes \mathrm{id}_A} C \otimes_{min} A \xrightarrow{p \otimes \mathrm{id}_A} D \otimes_{min} A \longrightarrow 0$$

is also exact.

Nuclearity of group C*-algebras and transformation group C*-algebras

In this section, we discuss under which conditions the group C*-algebras and the transformation group C*-algebras are nuclear.

For countable, discrete groups Γ, it is possible to clearly characterize when $C^*_r(\Gamma)$ is nuclear. A proof of the following result can be found in [BO08, Theorem 2.6.8].

Theorem 2.3.28. *Let Γ be a countable, discrete group. Then, the following are equivalent.*

- *(i)* Γ *is amenable,*
- *(ii)* $C^*_r(\Gamma)$ *is nuclear,*
- *(iii)* $C^*_r(\Gamma) \cong C^*(\Gamma)$ *canonically.*

For locally compact groups, the situation is more complicated. Indeed, for all connected groups G both $C^*(G)$ and $C^*_r(G)$ are nuclear (see [Pat88, p46]) even though G is obviously not necessarily amenable. However, the following follows from [LP91, Corollary 3.2].

Theorem 2.3.29. *Let G be a locally compact group G, then the following are equivalent.*

(i) G *is amenable,*

(ii) G *is inner amenable and* $C^*(G)$ *is nuclear.*

(iii) G *is inner amenable and* $C_r^*(G)$ *is nuclear.*

For transformation group C*-algebras, the following holds.

Theorem 2.3.30. *Let G be a locally compact group, X a locally compact space and $G \curvearrowright X$ a continuous action. Consider the following conditions.*

(i) $G \curvearrowright X$ *is topologically amenable,*

(ii) $C_0(X) \rtimes G \cong C_0(X) \rtimes_r G$ *canonically,*

(iii) $C_0(X) \rtimes G$ *is nuclear.*

(iv) $C_0(X) \rtimes_r G$ *is nuclear.*

Then, (i)⇒(ii)⇒(iii)⇒(iv). Moreover, if G is countable and discrete, then (i)⇔(ii)⇔(iii)⇔(iv).

The result for general locally compact groups can be found in [Ana02, Theorem 5.3]. The result for discrete groups is also due to Anantharaman-Delaroche and can be found in [Ana87, Théorème 4.5 and Proposition 4.8].

Exact groups and exactness of the group C*-algebra

The exactness of $C_r^*(G)$ is related to the following notion, which was introduced by Kirchberg and Wassermann in [KW99b].

Definition 2.3.31. A locally compact group G is called *exact* if the operation of taking the reduced crossed product preserves exactness, i.e. if for all C*-dynamical systems (A, G, σ_A), (B, G, σ_B) and (C, G, σ_C) and every G-equivariant short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the natural sequence

$$0 \longrightarrow A \rtimes_r G \longrightarrow B \rtimes_r G \longrightarrow C \rtimes_r G \longrightarrow 0$$

is also exact.

The corresponding property for full crossed products is always true (see for instance [BO08, p 173]).

As the name suggests, the following holds.

Theorem 2.3.32. *Let G be a locally compact group. If G is exact, then $C_r^*(G)$ is exact. If G is countable, discrete, then the converse is also true.*

The first part immediately follows from Theorem 2.3.27. The converse for discrete groups is due to [KW99b, Theorem 5.2]. The converse for locally compact groups is still open.

The class of exact groups is very large and contains among others all amenable groups, linear groups [GHW05] and hyperbolic groups [Ada94]. By [KW99a, Theorem 4.1 and Theorem 5.1] the class of exact groups is closed under taking closed subgroups and extensions. Examples of non-exact groups were given by Gromov [Gro03; AD08] and Osajda [Osa14].

The following result is due to Haagerup and Kraus (see [HK94, Theorem 2.1]) for countable, discrete groups and by Brodzki, Cave, and Li (see [BCL17, Corollary E]) for locally compact groups.

Theorem 2.3.33. *Every weakly amenable, locally compact group is exact.*

Recall that left equivariant Stone-Čech compactification $\beta^{lu}G$ is the spectrum of the algebra $C^{lu}(G)$ of bounded left uniformly continuous functions on G. The following result provides characterizations of exactness.

Theorem 2.3.34. *Let G be a locally compact group. Then, the following conditions are equivalent.*

(i) *G is exact,*

(ii) *G is amenable at infinity, i.e. G admits a continuous, amenable action of some compact space,*

(iii) *$G \curvearrowright \beta^{lu}G$ is amenable,*

(iv) *$C_b^{lu}(G) \rtimes_r G$ is nuclear.*

For countable, discrete groups, the characterizations (ii) and (iii) are due to Ozawa (see [Oza00, Theorem 3]), while characterization (iv) is due to Anantharaman-Delaroche (see [Ana87, Théorème 4.5]). For locally compact groups this result is due to Anantharaman-Delaroche (see [Ana02, Theorem 7.2]) and Brodzki, Cave, and Li (see [BCL17, Theorem A]).

2.4 Von Neumann algebras

In this section, we recall the definition, the basic properties and some standard examples of von Neumann algebras. Classical references for the notions we introduce are [AP14; Tak79; Tak03a; Bla06].

Definition 2.4.1. A *von Neumann algebra* is a unital $*$-subalgebra of $M \subseteq B(\mathcal{H})$ that is closed in the w.o. topology.

Due to a famous result of von Neumann, the above von Neumann algebras can also be characterized in a different way. Given a subset $S \subseteq B(\mathcal{H})$, we denote its *commutant* as the following set

$$S' = \{x \in B(\mathcal{H}) \mid xy = yx \text{ for every } y \in S\}.$$

The following result is due to von Neumann.

Theorem 2.4.2 (Bicommutant Theorem)**.** *Let $M \subseteq B(\mathcal{H})$ be a unital $*$-subalgebra. The following are equivalent.*

(i) *M is a von Neumann algebra, i.e. M closed in the w.o. topology.*

(ii) *M is closed in the s.o. topology.*

(iii) *$M'' = M$.*

By a result Sakai in [Sak56], von Neumann algebras can also be characterized abstractly as C*-algebras having a predual (i.e. there exists a Banach space F such that $F^* = M$). This motivates that we should view a von Neumann algebra as an object on itself, independent of the particular Hilbert space they are acting on. In particular, we say that two von Neumann algebras M and N are isomorphic if there exists a $*$-isomorphism $M \to N$. All notions we define will be independent of the concrete representation of the von Neumann algebra on a Hilbert space. A $*$-isomorphism $M \to M$ is called a $*$-automorphism. We denote by $\mathrm{Aut}(M)$ the set of all $*$-automorphisms $M \to M$.

Examples of von Neumann algebras include the whole of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and $L^\infty(X, \mu)$, the algebra of (essentially) bounded measurable function $X \to \mathbb{C}$, identified up to almost everywhere equality, where (X, μ) is a standard measure space. Here, we view $L^\infty(X, \mu)$ as a subalgebra of $B\big(L^2(X, \mu)\big)$ by identifying $f \in L^\infty(X, \mu)$ with the multiplication operator $L^2(X, \mu) \to L^2(X, \mu)$ defined by

$$(f\xi)(x) = f(x)\xi(x)$$

for all $\xi \in L^2(X,\mu)$ and a.e. $x \in X$. Moreover, one can prove that every abelian von Neumann algebra with separable predual is of this form. If $S \subseteq B(\mathcal{H})$ is a set, then the *von Neumann algebra generated by* S is $M = (S \cup S^*)''$. Note that M is indeed the smallest von Neumann algebra containing S.

Given a von Neumann algebra $M \subseteq B(\mathcal{H})$, a self-adjoint element $x \in M$ and a bounded measurable function f on the spectrum $\sigma(x)$, we define $f(x)$ by Borel functional calculus. Note that since $f(x)$ commutes with every element in M', we have that $f(x) \in M$. In particular, a von Neumann algebra contains all spectral projections of all elements. Moreover, it follows that von Neumann algebra is the $\|.\|$-closure of the linear span of all its projections.

Recall that a *partial isometry* is an operator $u \in B(\mathcal{H})$ such that u^*u (and hence uu^*) is a projection. We say the two projections $p, q \in M$ are *equivalent* and write $p \sim q$ if there exists a partial isometry $u \in M$ such that $p = u^*u$ and $q = uu^*$. Given two projections $p, q \in M$, we write $p \preceq q$ if there exists a projection $q' \leqslant q$ with $p \sim q'$. We say that a projection $p \in M$ is *infinite* if there exists a projection $p' \in M$ with $p' < p$ with $p \sim p'$. We say that p is *finite* if such a p' does not exist. We say the p is *minimal* if there exists no $p' \in M$ with $p' < p$. If M does not contain any minimal projection, then we say that M is *diffuse*.

Given a w.o. dense (or equivalently s.o. dense) $*$-subalgebra A of a von Neumann algebra $M \subseteq B(\mathcal{H})$, we can approximate every element in M by a net in A. The following theorem states that one can even approximate every such element by a $\|.\|$-bounded net in A. This is important, since, as we will see below, only the w.o. and s.o. topologies on bounded sets are intrinsic on a von Neumann algebra.

Theorem 2.4.3 (Kaplansky density theorem)**.** *Let $M \subseteq B(\mathcal{H})$ be von Neumann algebra and $A \subseteq M$ a w.o. dense $*$-subalgebra. Then,*

- *(a) the unit ball $(A)_1$ of A is strong* operator dense (and hence w.o. and s.o. dense) in the unit ball $(M)_1$ of M,*
- *(b) the unit ball $(A_{s.a.})_1$ of the self-adjoint part of A is s.o. dense (and hence w.o. dense) in the unit ball $(M_{s.a.})_1$ of the self-adjoint part of M.*

An important special type of von Neumann algebras is the following.

Definition 2.4.4. A *factor M* is a von Neumann algebra with trivial center, i.e. $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.

Factors can be seen as the 'building blocks' of von Neumann algebras, in the sense that every von Neumann algebra on a separable Hilbert space can be decomposed as a direct integral of factors (see for example [Bla06, Theorem III.1.6.3]).

We say the two factors M and N are *stably isomorphic* and write $M \cong_s N$ if there exists projections $p \in M$ and $q \in N$ such that $pMp \cong qNq$.

2.4.1 Normal linear functionals and maps

The class of linear functionals on von Neumann algebras that is the most interesting is the following class of so-called *normal* linear functionals. These are the functionals that are compatible with the additional structure of von Neumann algebras in the following sense.

Proposition 2.4.5. *Let M be a von Neumann algebra and $\omega : M \to \mathbb{C}$ a positive linear functional. Then, the following are equivalent.*

(i) *For any bounded increasing net $(x_i)_{i\in I}$ in M, we have*

$$\omega(\sup_i x_i) = \sup_i \omega(x_i).$$

(ii) *The restriction of ω to $(M)_1$, the unit ball of M, is w.o. continuous.*

(iii) *The restriction of ω to $(M)_1$, the unit ball of M, is s.o. continuous.*

If one (and hence all) of these conditions holds, we say that ω is normal.

Denote by M_*^+ the set of all normal positive linear functionals. Denote by M_* the space of all linear functionals on M that are w.o. continuous on the unit ball. We also call the functionals $\varphi \in M_*$ *normal*. Dixmier [Dix53] proved that every von Neumann algebra $M \cong (M_*)^*$, i.e. M_* is the (unique) predual of M. Since the unit ball of every Banach space is weak* dense in the unit ball of its double dual, this in particular implies that the set M_* of *normal* linear functionals on M is weak* dense in the set M^* of *all* linear functional on M. Moreover, the set M_*^+ of normal *positive* linear functionals is dense in the set all positive linear functionals M_+^* and that the set of normal states is dense in the set of all states.

Given a normal positive linear functional φ, we define its *support* as smallest projection s such that $\varphi(xs) = \varphi(x)$ for all $x \in M$. Given a partial isometry $u \in M$, we denote by $u \cdot \varphi$ the linear functional defined by $(u \cdot \varphi)(x) = \varphi(xu)$. The following result provides a decomposition of linear functionals into positive linear functionals. A proof can be found in [Tak79, Theorem III.4.2].

Theorem 2.4.6 (Polar decomposition of normal linear functionals)**.** *Let $\varphi : M \to \mathbb{C}$ be a normal linear functional on a von Neumann algebra M. Then,* *there exists a unique positive linear functional* $\psi : M \to \mathbb{C}$ *and a unique partial isometry* $u \in M$ *such that* $\psi = u^* \cdot \varphi$ *and* $\varphi = u \cdot \psi$*, and such that* u^*u *is the support of* ψ*. Moreover,* $\|\varphi\| = \|\psi\|$*.*

We denote the positive linear functional in the theorem above by $|\varphi|$.

By the uniqueness of u and $|\varphi|$, we have the following.

Theorem 2.4.7. *Let* $\varphi : M \to \mathbb{C}$ *be a normal linear functional and* $\beta \in \mathrm{Aut}(M)$*. Then,*

$$|\varphi \circ \beta| = |\varphi| \circ \beta.$$

Proof. Denoting $v = \beta^{-1}(x)$, we have $|\varphi| \circ \beta = v^* \cdot (\varphi \circ \beta)$ and $\varphi \circ \beta = v \cdot (|\varphi| \circ \beta)$. Moreover, v^*v is clearly the support of $|\varphi| \circ \beta$. □

In a similar fashion, *normal* positive linear maps $M \to N$ between two von Neumann algebras are maps that are compatible with all the available structure on a von Neumann algebra. A proof of the following can be found in [AP14, Proposition 2.5.8].

Proposition 2.4.8. *Let* $\Phi : M \to N$ *be a positive linear map. Then, the following conditions are equivalent.*

(i) *For any bounded increasing net* $(x_i)_{i \in I}$ *in* M*, we have*

$$\sup_i \Phi(x_i) = \Phi(\sup_i x_i).$$

(ii) *For every normal positive linear functional* ω *on* N*, the map* $\omega \circ \Phi$ *is a normal positive linear functional.*

(iii) *The restriction of* Φ *to the unit ball* $(M)_1$ *is continuous with respect to the w.o. topologies on* M *and* N*.*

If one of these conditions hold, we call Φ normal*.*

If Φ *is a* $*$*-morphism, the above conditions are also equivalent to*

(iv) *the restriction of* Φ *to the unit ball* $(M)_1$ *is continuous with respect to the s.o. topologies on* M *and* N*.*

The following two results illustrate that notion of isomorphism as defined above is indeed the natural one. A proof can be found in [AP14, Corollary 2.5.9].

Proposition 2.4.9. *Let M and N be von Neumann algebras. Then, every $*$-isomorphism $\alpha : M \to N$ is a normal positive linear map.*

From this result, it follows that the w.o. topology and the s.o. topology on the unit ball of a von Neumann algebra is independent of the Hilbert space on which M is represented. It is important to note that this is *not* true for the w.o. and s.o. topology on the whole of M. Therefore, we will only consider the restriction of the w.o. and s.o. topology to $\|.\|$-bounded sets as 'natural' topologies on von Neumann algebras.

The following result is very important, since it allows us identify a von Neumann algebra M with its image under faithful, normal, nondegenerate representations.

Proposition 2.4.10. *Let $\pi : M \to B(\mathcal{K})$ be a normal, nondegenerate representation of a von Neumann algebra M. Then, $\pi(M)$ is w.o. closed in $B(\mathcal{K})$ and is thus a von Neumann algebra.*

Unless stated otherwise, we will assume all von Neumann algebras to be *separable*, meaning that M can be represented faithfully on a separable Hilbert space, or equivalently that M has a separable predual.

2.4.2 Traces and the type classification of factors

Murray and von Neumann classified von Neumann factors into three types, based the structures of the projections in the von Neumann algebra. In this section, we will present these types in an alternative way using the existence of certain traces on the factor.

A factor M is said to be of *type I* if $M \cong B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If $\mathcal{H}$ is finite dimensional of dimension n, we say that $M \cong M_n(\mathbb{C})$ is of *type* I_n. If $\mathcal{H}$ is infinite dimensional, we say that M is of *type* I_∞. Note that there is only one factor of type I_n for each $n = 1, \dots, \infty$. One can prove that M is of type I if and only if it contains a minimal projection.

Recall that a *state* on a von Neumann algebra M is a positive linear functional $\varphi : M \to \mathbb{C}$ satisfying $\varphi(1) = 1$. The following is a generalization of the matrix trace on $M_n(\mathbb{C})$.

Definition 2.4.11. Let M be a von Neumann algebra. A state τ on M is called *tracial* if $\tau(xy) = \tau(yx)$ for all $x, y \in M$. A von Neumann algebra M is called *tracial* or *finite* if it admits a faithful normal tracial state.

The terminology finite is motivated by the fact that the unit 1 is a finite projection if and only if M is finite.

On a tracial von Neumann algebra (M, τ), one defines the following norm

$$\|x\|_2 = \tau(x^*x)$$

for $x \in M$. It is not hard to prove that the topology induced by $\|.\|_2$ coincides with the s.o. topology on $\|.\|$-bounded sets.

One can prove that if a factor admits a faithful normal tracial state, then it is unique (see for instance [AP14, Proposition 4.1.4]). Note that factors of type I_n for $n \in \mathbb{N}$ admit a tracial state, while the factor of type I_∞ does not. An infinite dimensional factor admitting such a faithful normal tracial state is said to be of *type II_1*.

Subalgebras of tracial von Neumann algebras admit the following special type of map.

Definition 2.4.12. Let M be a von Neumann algebra and B a von Neumann subalgebra. A *conditional expectation* from M to B is a positive linear map $E : M \to B$ satisfying

(i) $E(b) = b$ for all $b \in B$,

(ii) $E(b_1 x b_2) = b_1 E(x) b_2$ for all $b_1, b_2 \in B$ and all $x \in M$.

Theorem 2.4.13. *Let (M, τ) be a tracial von Neumann algebra and B a von Neumann subalgebra. Then, there exists a unique conditional expectation $E_B : M \to B$ satisfying $\tau \circ E_B = \tau$. Moreover, E_B is faithful and normal.*

A proof of this fact can for instance be found in [AP14, Theorem 9.1.2].

A more general notion of trace is the following. Recall that M_+ denotes the set of positive elements in M.

Definition 2.4.14. Let M be a von Neumann algebra. A *weight* is a map $\varphi : M_+ \to [0, +\infty]$ that is linear in the sense that $\varphi(\lambda x) = \lambda\varphi(x)$ and $\varphi(x+y) = \varphi(x) + \varphi(y)$ for any $\lambda \in \mathbb{R}^+$ and any $x, y \in M_+$. We say that φ is

- *faithful* if $\varphi(x) > 0$ for every nonzero $x \in M_+$.
- *normal* if $\varphi(\sup x_i) = \sup_i \varphi(x_i)$ for every bounded increasing net $(x_i)_i$ in M.

In the definition above, we used the convention that $0 \cdot (+\infty) = 0$.

Given a weight $\varphi : M^+ \to [0, +\infty]$, we define

$$\mathfrak{n}_\varphi = \{x \in M \mid \varphi(x^*x) < +\infty\}$$

$$\mathfrak{m}_\varphi = \Big\{ \sum_{i=1}^n x_i^* y_i \;\Big|\; x_i, y_i \in \mathfrak{n} \Big\}.$$

Using linearity and polarization, one proves that φ can be extended to a linear map $\varphi : \mathfrak{m}_\varphi \to \mathbb{C}$ (see [Tak03a, Definition VII.1.3]). Note that $\mathfrak{n}_\varphi$ is a left ideal of M. The subspace $\mathfrak{m}_\varphi$ is also called the *definition domain* of φ. In order to avoid trivial examples, one restricts to the following subclass of weights.

Definition 2.4.15. Let M be a von Neumann algebra. A weight $\varphi : M_+ \to [0, +\infty]$ is said to be *semifinite* if $\mathfrak{m}_\varphi$ is w.o. dense in M.

Now, a tracial weight is the following.

Definition 2.4.16. A weight $\mathrm{Tr} : M_+ \to \mathbb{C}$ on a von Neumann algebra is called *tracial* if $\mathrm{Tr}(x^*x) = \mathrm{Tr}(xx^*)$. A von Neumann algebra is called *semifinite* if it admits a faithful, normal, semifinite tracial weight Tr.

Note that all finite von Neumann algebras are also semifinite. Similarly as for tracial von Neumann algebras, we denote

$$\|x\|_{2,\mathrm{Tr}} = \mathrm{Tr}(x^*x)$$

for every $x \in M$. Note that $\|x\|_{2,\mathrm{Tr}} < +\infty$ if and only if $x \in \mathfrak{n}_{\mathrm{Tr}}$.

Similarly as for finite factors, if a factor M admits a faithful, normal, semifinite tracial weight, then it is unique up to scaling. We will denote this trace by Tr_M or Tr. The factor $B(\mathcal{H})$ of type I_∞ admits a faithful, normal, semifinite tracial weight defined by

$$\mathrm{Tr}(T) = \sum_{n\in\mathbb{N}} \langle Te_n, e_n \rangle ,$$

where $(e_n)_n$ is an orthonormal basis for $\mathcal{H}$. Other factors admitting a faithful, normal semifinite tracial weight Tr with $\mathrm{Tr}(\mathrm{id}_M) = +\infty$ are said to be of *type II_∞*. For a type II_∞ factor M and a projection $p \in M$ with $\mathrm{Tr}(p) < +\infty$, we have that pMp is a factor of type II_1 and that $M \cong pMp \mathbin{\overline{\otimes}} B(\mathcal{H})$, where $\overline{\otimes}$ denotes the von Neumann tensor product that we will introduce in Section 2.4.6 below. All factor not admitting any faithful, normal, semifinite tracial weight is said to be of *type III*.

Similar to tracial von Neumann algebras, also subalgebras of semifinite von Neumann algebras admit trace preserving conditional expectations under some conditions.

Theorem 2.4.17. *Let (M, Tr) be a semifinite von Neumann algebra and B a von Neumann subalgebra. If $\mathrm{Tr}|_B$ is semifinite, then there exists a unique conditional expectation $E_B : M \to B$ such that $\mathrm{Tr} \circ E_B = \mathrm{Tr}$. Moreover, E_B is faithful and normal.*

2.4.3 Tomita-Takesaki modular theory

In order to get a better understanding of type III factors, Tomita and Takesaki introduced their *modular theory* in [Tom67; Tak70]. We discuss the most important results and notions of this theory in this section. More complete references are [Haa88] and [Tak03a].

Let M be any von Neumann algebra and let φ be a normal, faithful, semifinite weight. Note that every von Neumann algebra admits such a weight (see for instance [Tak03a, Theorem VII.2.7]). We will construct a normal, faithful representation π_φ of M on a Hilbert space $\mathcal{H}_\varphi$ called the GNS-representation, after Gelfand, Naimark and Segal who introduced this construction.

Consider the ideal $\mathfrak{n}_\varphi = \{x \in M \mid \varphi(x^*x) < +\infty\}$ from the previous section. We equip it with the positive-definite, symmetric, sesquilinear form $\langle ., . \rangle_\varphi$ defined by

$$\langle x, y \rangle_\varphi = \varphi(y^*x)$$

for any $x, y \in \mathfrak{n}_\varphi$. (Note that we use here that φ extends to a linear map on the definition domain $\mathfrak{m}_\varphi$.) Let $L^2(M, \varphi)$ be the completion of $\mathfrak{n}_\varphi$ with respect to this sesquilinear form. For the sake of clarity, we will denote an element of $x \in \mathfrak{n}_\varphi$ by $\eta_\varphi(x)$ when viewed as an element in $L^2(M, \varphi)$. This way η_φ becomes an injection $\mathfrak{n}_\varphi \hookrightarrow L^2(M, \varphi)$.

For every $x \in M$, we have

$$\|\eta_\varphi(xy)\|_\varphi^2 = \varphi(y^*x^*xy) \leqslant \|x\|^2 \|\eta_\varphi(y)\|_\varphi$$

and hence $\eta_\varphi(y) \mapsto \eta_\varphi(xy)$ is bounded and hence extends to an operator $\pi_\varphi(x)$ on $L^2(M, \varphi)$. This yields a faithful, nondegenerate representation $\pi_\varphi : M \to B\big(L^2(M, \varphi)\big)$. This representation is called the *GNS-representation* or *semicyclic* representation of M with respect to φ.

Now, the involution on M defines a (usually unbounded) closable, conjugate linear operator $S_0 : \eta_\varphi(x) \mapsto \eta_\varphi(x^*)$ on the dense subset $\eta_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*) = \{\eta_\varphi(x) \mid \varphi(x^*x) < +\infty \text{ and } \varphi(xx^*) < +\infty\}$ of $\mathcal{H}_\varphi$. We denote the closure of this operator by S_φ. The *modular operator* is defined as $\Delta_\varphi = S_\varphi^* S_\varphi$. Note that Δ_φ is self-adjoint and positive. Let $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ be the polar decomposition. Then $J_\varphi : L^2(M, \varphi) \to L^2(M, \varphi)$ is an anti-unitary operator called the *modular conjugation*.

The following is the main result in Tomita-Takesaki theory. A proof can for instance be found in [Tak03a, Theorem VI.1.19].

Theorem 2.4.18. *Let M be a von Neumann algebra and φ a faithful, normal, semifinite weight on M. Then,*

$$J_\varphi \pi_\varphi(M) J_\varphi = \pi_\varphi(M)' \qquad \textit{and} \qquad \Delta_\varphi^{it} \pi_\varphi(M) \Delta_\varphi^{-it} = \pi_\varphi(M)$$

for all $t \in \mathbb{R}$.

Identifying M with $\pi_\varphi(M)$, the above yields a (s.o. continuous) group of automorphisms $\{\sigma_t^\varphi\}_{t\in\mathbb{R}}$ given by

$$\sigma_t^\varphi(x) = \Delta^{it} x \Delta^{-it}$$

for $t \in \mathbb{R}$ and $x \in M$. This group is called the *modular automorphism group* of M with respect to φ. One has $\varphi \circ \sigma_t^\varphi = \varphi$ for every $t \in \mathbb{R}$.

The set

$$M^\varphi = \{x \in M \mid \sigma_t^\varphi(x) = x\}$$

is called the *centralizer* of M with respect to φ. One can prove that $x \in M^\varphi$ if and only if $x\mathfrak{m}_\varphi \subseteq \mathfrak{m}_\varphi$ and $\mathfrak{m}_\varphi x \subseteq \mathfrak{m}_\varphi$ and $\varphi(xy) = \varphi(yx)$ for all $y \in \mathfrak{m}_\varphi$. A proof of this fact can be found in [Tak03a, Theorem 2.6].

The modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ does depend on the choice of weight φ. However, due to Connes' Cocycle Derivative Theorem (see [Tak03a, Theorem VIII.3.3]), they are related in the following way: if ψ is another normal, faithful, semifinite weight on M, then there exists an s.o. continuous one-parameter family of unitaries $(u_t)_{t\in\mathbb{R}}$ in M satisfying $u_{t+s} = u_t\sigma_t^\varphi(u_s)$ and $\sigma_t^\varphi(x) = u_t\sigma_t^\psi(x)u_t^*$ for all $s, t \in \mathbb{R}$ and all $x \in M$. Moreover, under certain additional conditions, this one-parameter family is unique.

Note that in the case that M is semifinite and $\varphi = \mathrm{Tr}$ is a faithful, normal, semifinite trace, we have that S_φ itself is bounded and anti-unitary and hence $S_\varphi = J_\varphi$ and $\Delta_\varphi = \mathrm{id}$ and $\sigma_t = \mathrm{id}_M$. It follows from Connes' Cocycle Derivative Theorem that a von Neumann algebra M with a faithful, normal, semifinite weight φ admits a faithful, normal, *tracial*, semifinite weight if and only if $\sigma_t^\varphi = \mathrm{Ad}(u_t)$ for some s.o. continuous one-parameter family $\{u_t\}_{t\in\mathbb{R}}$ in M.

The following generalizes Theorems 2.4.13 and 2.4.17. A proof can be found in [Tak03a, Theorem IX.4.2].

Theorem 2.4.19. *Let M be a von Neumann algebra with a faithful, normal weight φ. Let B be a von Neumann subalgebra such that $\varphi|_B$ is semifinite. Then, there exists a conditional expectation $E_B : M \to B$ satisfying $\varphi \circ E_B = \varphi$ if and only if $\sigma_t^\varphi(B) = B$ for all $t \in \mathbb{R}$. In that case, E_B is the unique conditional expectation satisfying $\varphi \circ E_B = \varphi$. Moreover, it is automatically faithful and normal.*

Definition 2.4.20. Let M be a von Neumann algebra. A von Neumann subalgebra $B \subseteq M$ is said to be *with expectation* if there exists a faithful, normal conditional expectation $E : M \to B$.

Many constructions of von Neumann subalgebras out of subalgebras with expectation remain with expectation. For instance, given an inclusion of von Neumann algebras $B \subseteq M$, the *normalizer* is defined as

$$\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}.$$

Similarly, the *stable normalizer* is

$$\mathcal{N}_M^{\mathrm{s}}(A) = \{x \in M \mid xAx^* \subseteq A \text{ and } x^*Ax \subseteq A\}.$$

The following can easily be proved using Theorem 2.4.19. For the readers convenience, we include a full proof.

Proposition 2.4.21. *Let M be a von Neumann algebra and B a von Neumann subalgebra with expectation. Then,*

- *(a) the subalgebra $B' \cap M$ is with expectation,*
- *(b) the subalgebras $\mathcal{N}_M(B)''$, $\mathcal{N}_M^s(B)''$ are with expectation,*

Proof. Denote by $E : M \to B$ the faithful, normal, conditional expectation. Fix a faithful, normal state φ on B. We still denote by φ its extension $\varphi \circ E$ to M. We denote by $\{\sigma_t\}_{t\in\mathbb{R}}$ the modular automorphism group with respect to φ. Note that by Theorem 2.4.19 we have $\sigma_t(B) = B$ for all $t \in \mathbb{R}$.

If $x \in B' \cap M$ and $b \in B$, then

$$\sigma_t(x)\sigma_t(b) = \sigma_t(xb) = \sigma_t(bx) = \sigma_t(b)\sigma_t(x)$$

for all $t \in \mathbb{R}$. Hence, $\sigma_t(B'\cap M) = B'\cap M$ and by Theorem 2.4.19 the subalgebra $B' \cap M$ is with expectation.

Similarly, one has $\sigma_t\big(\mathcal{N}_M(A)\big) = \mathcal{N}_M(A)$ and $\sigma_t\big(\mathcal{N}_M^{\mathrm{s}}(A)\big) = \mathcal{N}_M^{\mathrm{s}}(A)$. Hence, also $\sigma_t(P) = P$ and $\sigma_t(Q) = Q$ for $P = \mathcal{N}_M(A)''$ and $Q = \mathcal{N}_M^{\mathrm{s}}(A)''$. Again applying Theorem 2.4.19 yields the result. □

The standard representation

We briefly return to the GNS-representation on $L^2(M, \varphi)$ above. Let again M be a von Neumann algebra and φ a faithful, normal, semifinite weight on M.

The GNS-Hilbert space $L^2(M, \varphi)$ contains a *positive cone* P that is defined as the closure of

$$\left\{ \eta_\varphi(xx^*) \;\middle|\; x \in \bigcup_{n \in \mathbb{Z}} \mathrm{dom}(\Delta^n) \text{ and } \Delta^n x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^* \text{ for all } n \in \mathbb{Z} \right\}.$$

This cone is self-dual in the sense that

$$P_\varphi = \{\xi \in \mathcal{H}_\varphi \mid \langle \xi, \eta \rangle \geqslant 0 \text{ for all } \eta \in P_\varphi\}.$$

Identifying M with its image under the GNS-representation, the tuple $(M, \mathcal{H}_\varphi, J_\varphi, P_\varphi)$ satisfies the following properties.

Theorem 2.4.22. *Let M be a von Neumann algebra and φ a faithful, normal, semifinite weight on M. Let $\mathcal{H}_\varphi$, J_φ and P_φ be as above. Then,*

(a) $J_\varphi M J_\varphi = M'$

(b) $J_\varphi x J_\varphi = x^*$ *for* $x \in \mathcal{Z}(M)$

(c) $J_\varphi \xi = \xi$ *for all* $\xi \in P_\varphi$

(d) $x J_\varphi x J_\varphi P_\varphi \subseteq P_\varphi$ *for all* $x \in M$

Proofs of this theorem can be found in [Haa75, Theorem 1.6] and [Tak03a, Theorem IX.1.2]. Tuples satisfying these properties are given a special name.

Definition 2.4.23. Let M be a von Neumann algebra. A tuple $(M, \mathcal{H}, J, P)$, where M is a von Neumann algebra represented on the Hilbert space $\mathcal{H}$, J an anti-unitary operator on $\mathcal{H}$ and P a self-dual cone, is said to be a *standard form* of M if

(i) $JMJ = M'$

(ii) $JxJ = x^*$ for $x \in \mathcal{Z}(M)$

(iii) $J\xi = \xi$ for $\xi \in P$

(iv) $xJxJP \subseteq P$ for all $x \in M$

The standard form of a von Neumann algebra is unique in the following sense.

Theorem 2.4.24. *Suppose that $(M_1, \mathcal{H}_1, J_1, P_1)$ and $(M_2, \mathcal{H}_2, J_2, P_2)$ are standard forms of M_1 and M_2 respectively and that $\pi : M_1 \to M_2$ is an isomorphism. Then, there is a unique unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$\pi(x) = UxU^*, \qquad J_2 = UJ_1U^* \qquad \text{and} \qquad P_2 = UP_1$$

for all $x \in M_1$.

Proofs can be found in [Haa75, Theorem 2.3] and [Tak03a, Takesaki IX.1.14].

Applying this result to $\Phi \in \mathrm{Aut}(M)$ yields the following. See also [Haa75, Theorem 3.2] and [Tak03a, Takesaki IX.1.15].

Theorem 2.4.25. *Let $(M, \mathcal{H}, J, P)$ be a standard form and $\Phi \in \mathrm{Aut}(M)$. Then, there exists a unique unitary $U_\Phi \in \mathcal{U}(\mathcal{H})$ satisfying*

$$\Phi(x) = U_\Phi x U_\Phi^*, \qquad J = U_\Phi J U_\Phi^* \qquad \text{and} \qquad P = U_\Phi P.$$

Moreover, the map $\Phi \mapsto U_\Phi$ is an isomorphism of $\mathrm{Aut}(M)$ to the group of all unitaries in $\mathcal{U}(\mathcal{H})$ satisfying UMU^*, $U = UJU^*$ and $P = UP$.

The unitary U in the theorem above is called the *canonical unitary implementing* Φ.

The following result says that every normal (positive) linear functional can be implemented by vectors of the positive cone of the standard form.

Theorem 2.4.26. *Let $(M, \mathcal{H}, J, P)$ be a standard form.*

(a) For every normal positive linear functional $\varphi \in M_*^+$, there is a unique $\xi \in P$ such that

$$\varphi(x) = \langle x\xi, \xi \rangle ,$$

for all $x \in M$.

(b) For every normal linear functional $\varphi \in M_*$, there exist $\xi, \eta \in \mathcal{H}$ such that

$$\varphi(x) = \langle x\xi, \eta \rangle$$

Denoting $\omega_\xi(x) = \langle x\xi, \xi \rangle$, we have the following result.

Theorem 2.4.27 (Powers-Størmer inequality)**.** *Let $(M, \mathcal{H}, J, P)$ be a standard form. For any $\xi, \eta \in P$, we have*

$$\|\xi - \eta\|^2 \leqslant \|\omega_\xi - \omega_\eta\| \leqslant \|\xi - \eta\| \, \|\xi + \eta\| .$$

Again, proofs of the previous two results can be found in [Tak03a, Theorem IX.1.2] and [Haa75, Lemma 2.10] (along with Theorem 2.4.6).

2.4.4 Group von Neumann algebras

The group von Neumann algebra is the von Neumann algebraic analogue of the (reduced) group C*-algebra introduced in Section 2.3.2. A reference for this construction is [Tak03a, Section VII.3].

Let G be a locally compact group. Recall the *left* and the *right regular representation* $\lambda : G \to \mathcal{U}(L^2(G))$ and $\rho : G \to \mathcal{U}(L^2(G))$ defined by

$$(\lambda_g \xi)(t) = \xi(g^{-1}t) \qquad \text{and} \qquad (\rho_g \xi)(t) = \delta_G(g)^{1/2}\xi(tg)$$

for $\xi \in L^2(G)$. We define the group von Neumann algebra as follows.

Definition 2.4.28. The *(left) group von Neumann algebra* $L(G)$ of a locally compact group G is the von Neumann algebra generated by $\{\lambda_g\}_{g\in G}$, i.e.

$$L(G) = \overline{\operatorname{span}\{\lambda_g\}_{g\in G}}^{\text{w.o.}}.$$

It is conventional to denote the elements $\lambda_g \in L(G)$ by u_g. Similarly, the right group von Neumann algebra is defined as

$$R(G) = \overline{\operatorname{span}\{\rho_g\}_{g\in G}}^{\text{w.o.}}.$$

Obviously, we have $L(G) \subseteq R(G)'$ and $R(G) \subseteq L(G)'$.

As before, we define

$$\lambda(f) = \int_G f(t)\lambda_t \,\mathrm{d}t \qquad \text{and} \qquad \rho(f) = \int_G f(t)\rho_t \,\mathrm{d}t$$

for $f \in C_c(G)$. Note that the action of $\lambda(f)$ and $\rho(f)$ on $B\big(L^2(G)\big)$ are given by (2.3.4) and (2.3.5). By approximating a function $f \in C_c(G)$ by step functions, one constructs a sequence in $\operatorname{span}\{u_g\}_{g\in G}$ converging to $\lambda(f) = \int_G f(g)\lambda_g$ in the w.o. topology. Hence, $\{\lambda(f)\}_{f\in C_c(G)} \subseteq L(G)$ and similarly $\{\rho(f)\}_{f\in C_c(G)} \subseteq R(G)$. Moreover, by [Tak03a, Proposition 3.1] one even has the following.

Theorem 2.4.29. *Let G be a locally compact group. Then,*

$$L(G) = \overline{\{\lambda(f)\}_{f\in C_c(G)}}^{w.o.} \qquad \textit{and} \qquad R(G) = \overline{\{\rho(f)\}_{f\in C_c(G)}}^{w.o.}.$$

In particular, the reduced group C^*-algebra $C^*_r(G)$ is a w.o. dense $*$-subalgebra of $L(G)$.

By the same theorem, one also has that the left and right group von Neumann algebras are each others commutant.

Theorem 2.4.30. *Let G be a locally compact group. Then,*

$$L(G) = R(G)' \qquad \textit{and} \qquad R(G) = L(G)'.$$

The von Neumann algebra $L(G)$ carries a natural normal, faithful, semifinite weight φ_G, called the *Plancherel weight* (see [Tak03a, Theorem 3.4]). It satisfies

$$\varphi_G\big(\lambda(f)\big) = f(e)$$

for all $f \in C_c(G)$. Defining $f * g$ and g^* as in (2.3.2) and (2.3.3), we have

$$\varphi_G\big(\lambda(f)^*\lambda(g)\big) = (f^* * g)(e) = \langle f, g\rangle$$

for all $f, g \in C_c(G)$. Since $C_c(G)$ is dense in $L^2(G)$, it follows that $L^2(G)$ is the GNS-Hilbert space of the weight φ_G. The associated modular operator Δ_G is given by

$$(\Delta_G \xi)(t) = \xi(t)\delta_G(t)$$

for $\xi \in \operatorname{dom} \Delta$, where

$$\operatorname{dom} \Delta_G = \left\{ \xi \in L^2(G) \;\middle|\; \int_G \delta_G(g)^2 |\xi(g)|^2 < +\infty \right\}.$$

The modular conjugation J is given by

$$(J\xi)(t) = \overline{\xi(t)}\delta_G(t)^{-1/2}. \tag{2.4.1}$$

for $\xi \in L^2(G)$ and $t \in G$, and the modular automorphism group satisfies

$$\sigma_t\big(u_g\big) = \delta_G(g)^{it} u_g \tag{2.4.2}$$

for every $g \in G$ and $t \in \mathbb{R}$. With $P = L^2(G)^+$, the tuple $\big(L(G), L^2(G), J, P\big)$ is a standard representation for $L(G)$. Note that in particular φ_G is a trace if and only if G is unimodular. Also note that

$$J u_t J = \rho_t \qquad \text{and} \qquad J\lambda(f)J = \rho(\overline{f})$$

for $f \in C_c(G)$.

If Γ is countable and discrete, then one can characterize when $L(\Gamma)$ is a factor. A proof can be found in [AP14, Proposition 1.3.9].

Theorem 2.4.31. *Let Γ be a countable, discrete group. Then, $L(\Gamma)$ is a factor if and only if Γ is* icc*, i.e. every nontrivial conjugacy class of Γ is infinite. In that case, Γ is always a factor of type II_1.*

For locally compact groups, the situation is much more complicated and such a nice characterization does not exist. Moreover, for many groups 'natural' groups, the group von Neumann algebra is amenable (see Section 2.4.8 below) or even type I. For instance, the group von Neumann algebra of a connected locally compact group is always amenable by [Con76, Corollary 6.9] and $L\big(\mathrm{SL}_n(\mathbb{Q}_p)\big)$ is type I by [Ber74].

However, for each of the types I_∞, II_∞ and III_λ for $0 \leqslant \lambda \leqslant 1$ (see Section 2.4.5 below) there are examples of (nondiscrete) locally compact groups whose group von Neumann algebra $L(G)$ is a factor of this type. There exist both examples for which these group von Neumann algebras are amenable and examples for which they are not (see [Sut78, Section 5]).

The Fourier algebra and the group von Neumann algebra

Recall from Section 2.1.2 that the Fourier algebra $A(G)$ is defined as the algebra of coefficients of the left regular representation. Since every normal linear functional $\varphi \in L(G)_*$ is of the form $x \mapsto \langle x\xi, \eta\rangle$ for some $\xi, \eta \in L^2(G)$, and since $\{u_g\}_{g\in G}$ is w.o. dense in $L(G)$, it follows that the map $L(G)_* \to A(G)$ defined by sending a $\varphi \in L(G)_*$ to the function $g \mapsto \varphi(u_g)$ is a well-defined bijective linear isometry.

Using this identification, one proves the following result. See [dCH85, Proposition 1.2] for details.

Proposition 2.4.32. *Let $\varphi : G \to \mathbb{C}$ be a function on a locally compact group G. Then, the following are equivalent.*

- *φ is a Fourier multiplier*
- *There exists a (unique) w.o. continuous map $\Phi_\varphi : L(G) \to L(G)$ satisfying $\Phi_\varphi(u_g) = \varphi(g)u_g$*

The map Φ in the proposition above is actually the dual of the map $m_\varphi : A(G) \to A(G)$, using the identification $A(G) \cong L(G)_*$ above.

The name completely bounded multipliers is motivated by the following result. The result was originally proven by [BF84]. A more recent self-contained proof can be found in [Haa16, Theorem 3.2].

Theorem 2.4.33. *A Fourier multiplier φ is completely bounded in the sense of Definition 2.1.4 if and only if its induced map $\Phi_\varphi : L(G) \to L(G)$ is completely bounded.*

2.4.5 Crossed products

There is also a version of the crossed product construction for von Neumann algebras. We will discuss this construction in this section. The interested reader can find more details in [Tak03a, Chapter X], [Bla06, Section III.3.2] and [Sau77].

Let N be a von Neumann algebra and $G \curvearrowright^\alpha N$ an action by automorphisms. We say that $G \curvearrowright^\alpha N$ is continuous in the pointwise s.o. topology if the associated group morphism $\sigma : G \to \mathrm{Aut}(N)$ is continuous for this topology, i.e.

$$\lim_{g\to h} \|\alpha_g(x)\xi - \alpha_h(x)\xi\| = 0$$

for every $x \in N$ and every $\xi \in \mathcal{H}$. Note that the pointwise s.o. topology on $\mathrm{Aut}(N)$ coincides with the pointwise w.o. topology and the pointwise strong* operator topology. The analogue of a C*-dynamical system is now the following.

Definition 2.4.34. A *von Neumann dynamical system* or a *von Neumann covariant system* is a triple (G, N, α) consisting of a locally compact group G, a von Neumann algebra N and an action $G \curvearrowright^{\alpha} N$ by automorphisms that is continuous in the pointwise s.o. topology.

Fix a von Neumann dynamical system (G, N, α) and suppose that $N \subseteq B(\mathcal{H})$. Similarly as with the reduced crossed product, we define a covariant representation (π_α, u) of this dynamical system on $\mathcal{K} = L^2(G) \otimes \mathcal{H} \cong L^2(G; \mathcal{H})$ by

$$\big(\pi_\alpha(x)\xi\big)(t) = \alpha_t^{-1}(x)\xi(t) \qquad \text{and} \qquad (u_g\xi)(t) = \xi(g^{-1}t)$$

for $x \in M$ and $g, t \in G$. The crossed product is now the following.

Definition 2.4.35. The *crossed product* $N \rtimes G$ of the von Neumann dynamical system (N, G, α) is the von Neumann algebra generated by the operators $\{u_g\}_{g \in G}$ and $\{\pi_\alpha(x)\}_{x \in N}$, i.e.

$$N \rtimes G = \overline{\operatorname{span}\{u_g \pi_\alpha(x)\}_{g \in G, x \in N}}^{\text{w.o.}}.$$

As is to be expected, the crossed product is independent of the chosen normal, faithful representation of N above (see [Tak03a, Theorem X.1.7]). In what follows, we will identify N with its inclusion in the crossed product $M = N \rtimes G$. Note that the group von Neumann algebra $L(G)$ is just to the crossed product where $N = \mathbb{C}$.

Denote by $\mathcal{K}(G; N)$ the algebra of compactly supported functions $f : G \to N$ that are continuous on the strong* operator topology. For $f \in \mathcal{K}(G; N)$ we put

$$(\pi_\alpha \rtimes u)(f) = \int_G u_t \pi_\alpha\big(f(t)\big)\, \mathrm{d}t.$$

The following is [Tak03a, Lemma X.1.8].

Theorem 2.4.36. *Let (G, N, σ) be a von Neumann dynamical system. Then,*

$$N \rtimes G = \overline{\{(\pi_\alpha \rtimes u)(f)\}_{f \in \mathcal{K}(G;N)}}^{w.o.}.$$

Moreover, one can prove that every von Neumann dynamical system (G, N, σ) admits a w.o. dense C*-subalgebra $\mathcal{N} \subseteq N$ such that $G \curvearrowright \mathcal{N}$ is point-norm continuous and hence $(G, \mathcal{N}, \sigma)$ is a C*-dynamical system. It follows that $\mathcal{N} \rtimes_r G$ is a w.o. dense C*-subalgebra in $N \rtimes G$ (see [Bla06, Theorems III.3.2.4 and III.3.2.7]).

Defining multiplication and involution on $\mathcal{K}(G; N)$ by

$$(f * g)(t) = \int_G \alpha_s\big(f(ts)\big) g(s^{-1})\,\mathrm{d}t \qquad \text{and} \qquad f^*(t) = \delta_G(t)^{-1} \alpha_t^{-1}\big(f(t^{-1})^*\big)$$

for $f, g \in \mathcal{K}(G; N)$ and $t \in G$, the map $\pi_\alpha \rtimes u$ becomes an injective $*$-morphism. Identifying $\mathcal{K}(G; N)$ with its image under $\pi_\alpha \rtimes u$ we have

$$(f\xi)(t) = \int_G \alpha_s\big(f(ts)\big)\xi(s^{-1})\,\mathrm{d}s \tag{2.4.3}$$

for $f \in \mathcal{K}(G; N)$, $\xi \in \mathcal{K}$ and $t \in G$.

Let φ be a weight on N. Then, one can construct a so-called *dual weight* $\tilde{\varphi}$ on the crossed product $M = N \rtimes G$ (see [Tak03a, Theorem X.1.17]). It satisfies

$$\tilde{\varphi}(f) = \varphi\big(f(e)\big)$$

for every $f \in \mathcal{K}(G; N)$.

Fixing a standard representation $(N, \mathcal{H}_\varphi, J_\varphi, P_\varphi)$ for N, the crossed product is in standard representation on $\mathcal{K} = L^2(G) \otimes \mathcal{H}_\varphi$, where the modular conjugation J is given by

$$(J\xi)(t) = \delta_G(t)^{-1/2} U_\varphi(t)^* J_\varphi \xi(t^{-1}),$$

for $\xi \in \mathcal{K}$ and a.e. $t \in G$, where $U_\varphi(t) \in \mathcal{U}(\mathcal{H})$ denotes the canonical unitary implementation of the automorphism $\alpha_t^\varphi \in \mathrm{Aut}(N)$ (see Theorem 2.4.25). The modular automorphism group is given by

$$\sigma_t^{\tilde{\varphi}}(x) = \sigma_t^{\varphi}(x) \qquad \text{and} \qquad \sigma_t^{\tilde{\varphi}}(u_g) = \delta_G(g)^{it} u_g [D(\varphi \circ \alpha_g) : D\varphi]_t, \tag{2.4.4}$$

for $x \in N$, $g \in G$ and $t \in \mathbb{R}$. Here, $[D(\varphi \circ \alpha_g) : D\varphi]_t$ denotes the Connes cocycle derivative of $\varphi \circ \alpha_g$ with respect to φ. See [Tak03a, Lemma X.1.13 and Theorem X.1.17] for proofs of these facts.

The group measure space construction

As with the crossed product for C*-algebras, the case that the base algebra N is abelian is an interesting special case. Recall that every abelian von Neumann algebra is of the form $L^\infty(X, \mu)$ for some measure space μ. Moreover, by [Mac62], every pointwise s.o. continuous action is induced by a measurable action $G \curvearrowright (X, \mu)$.

We work with the following 'well-behaved' measure spaces.

Definition 2.4.37. A *standard Borel space* X is a complete, separable metric space X equipped with the Borel σ-algebra. If μ is a measure on X, then we call the pair (X, μ) a *standard measure space.* If μ is a probability measure, then we call (X, μ) a *standard probability space.*

One can prove that all standard Borel spaces are either countable (equipped with the discrete σ-algebra), or isomorphic to $\mathbb{R}$ equipped with the typical Borel σ-algebra. All standard probability spaces (X, μ) are isomorphic to the interval $[0, 1]$ equipped with the Lebesgue measure, a discrete measure, or a convex combination of both. In the rest of this thesis, we will assume all measure spaces to be standard measure spaces. Moreover, we will assume all measures to be σ-finite.

Given two measure spaces (X_1, μ_1) and (X_2, μ_2), a *Borel isomorphism* $\theta : X_1 \to X_2$ is a bijective bimeasurable map. We say that θ is *nonsingular* if θ maps nullsets to nullsets. We say that θ is *measure preserving (mp)* if $\theta_* \mu_1 = \mu_1 \circ \theta^{-1} = \mu_2$. We say that θ is *probability measure preserving (pmp)* if it is measure preserving and μ_i are probability measures.

For actions of groups on measure spaces, we have the following terminology.

Definition 2.4.38. Let G be a locally compact group and (X, μ) a standard measure space. An action $G \curvearrowright (X, \mu)$ is called *Borel* if the map $G \times X \to X : (g, x) \mapsto gx$ is Borel. We say that

(a) $G \curvearrowright (X, \mu)$ is *nonsingular* if $\mu(gE) = 0$ for all $g \in G$ and all Borel sets $E \subseteq X$ with $\mu(E) = 0$,

(b) $G \curvearrowright (X, \mu)$ is *measure preserving (mp)* if $\mu(gE) = \mu(E)$ for all $g \in G$ and all Borel sets $E \subseteq G$,

(c) $G \curvearrowright (X, \mu)$ is *probability measure preserving (pmp)* if it is measure preserving and μ is a probability measure.

A nonsingular action $G \curvearrowright (X, \mu)$ on a standard measure space induces a s.o. continuous action α of G on $A = L^\infty(X, \mu)$ by $*$-automorphisms given by

$$\alpha_g(f)(x) = f(g^{-1}x)$$

for all $f \in A$, $g \in G$ and a.e. $x \in X$. The covariant representation (π_α, u) of (A, G, α) on $\mathcal{K} = L^2(G) \otimes L^2(X)$ as above is then given by

$$(\pi_\alpha(f)\xi)(t, x) = f(tx)\xi(t, x) \qquad \text{and} \qquad (u_g\xi)(t, x) = \xi(g^{-1}t, x)$$

for $\xi \in \mathcal{K}$, $f \in A$, $g \in G$, a.e. $t \in G$ and a.e. $x \in X$.

Let

$$M = A \rtimes G = \overline{\operatorname{span}\{u_g \pi_\alpha(f)\}_{g\in G, f\in A}}^{\text{w.o.}}$$

be the crossed product. The measure μ induces a natural normal, semifinite (tracial) weight φ given by

$$\varphi(f) = \int_X f(x)\,\mathrm{d}\mu.$$

Since $\big(A, L^2(X), J_1, L^2(X)^+\big)$ is a standard representation, where J_1 is given by taking pointwise complex conjugation, we have that M is in standard representation on $\mathcal{K}$, where the modular conjugation is given by

$$(J\xi)(t,x) = \delta_G(t)^{-1/2}\left(\frac{\mathrm{d}(t^{-1}\cdot\mu)}{\mathrm{d}\mu}(x)\right)^{1/2}\overline{\xi(t^{-1}, tx)}.$$

Again, identifying $f \in A$ with the operator $\pi_\alpha(f) \in M$, the modular automorphism group with respect to the dual weight $\tilde{\varphi}$ is given by

$$\sigma_t^{\tilde{\varphi}}(f) = f \qquad \text{and} \qquad \sigma_t^{\tilde{\varphi}}(u_g) = \delta_G(g)^{it} u_g \left(\frac{\mathrm{d}(g^{-1}\cdot\mu)}{\mathrm{d}\mu}\right)^{it}$$

We will now describe conditions on the action $G \curvearrowright (X,\mu)$ under which the crossed product is a factor. For this, we need the following terminology.

Definition 2.4.39. Let G be a locally compact group and (X,μ) a standard measure space. Suppose that $G \curvearrowright (X,\mu)$ is a nonsingular action. We say that

(a) the action $G \curvearrowright (X,\mu)$ is *free* if $gx \neq x$ for all $x \in X$ and all $g \in G\setminus\{e\}$,

(b) the action $G \curvearrowright (X,\mu)$ is *essentially free* if

$$\{x \in X \mid \exists g \in G\setminus\{e\} : gx = x\}$$

is a null set.

(c) the action $G \curvearrowright (X,\mu)$ is *ergodic* if every measurable G-invariant subset $A \subseteq G$ satisfies $\mu(A) = 0$ or $\mu(G\setminus A) = 0$

Note that the set $\{x \in X \mid \exists g \in G\setminus\{e\} : gx = x\}$ in the definition above is Borel by [MRV13, Lemma 10].

Sauvageot proved the following result in [Sau77, Proposition 2.1].

Theorem 2.4.40. *Let $G \curvearrowright (X,\mu)$ be a nonsingular action of a locally compact group on a standard measure space. Denote $A = L^\infty(X,\mu)$ and let $M = A \rtimes G$. If $G \curvearrowright (X,\mu)$ is essentially free, then*

(a) *A is a* maximal abelian subalgebra (MASA), *i.e.* $A' \cap M = A$,

(b) $\mathcal{Z}(M) = A^G$, *where* $A^G = \{f \in A \mid \alpha_g(f) = f\}$.

In particular, if $G \curvearrowright (X,\mu)$ is essentially free and ergodic, then M is a factor.

It is important to note that M can be a factor while $G \curvearrowright (X,\mu)$ is not essentially free. Indeed, the group von Neumann algebra can be a factor, while the action of G on a single point is never free.

In the case of discrete groups, the type of the crossed product can be characterized on the level of the action. A proof of this result can be found in [Tak79, Theorem V.7.12].

Theorem 2.4.41. *Let G be a countable, discrete group and $G \curvearrowright (X,\mu)$ be a nonsingular action on a standard measure space. Suppose that $G \curvearrowright (X,\mu)$ is essentially free and ergodic. Let $M = L^\infty(X,\mu) \rtimes G$, then*

(a) M is of type I if and only if X contains an atom $x_0 \in X$,

(b) M is of type II_1 if and only if X contains no atoms and there exists an invariant probability measure ν in the same measure class as μ,

(c) M is of type II_∞ if and only if X contains no atoms and there exists an invariant measure ν in the same measure class as μ that is not finite,

(d) M is of type III if and only if X contains no atoms and there exists no invariant measure in the same measure class as μ.

For locally compact groups, Sauvageot proved the following theorem in [Sau77, Proposition 3.3].

Theorem 2.4.42. *Let $G \curvearrowright (X,\mu)$ be a nonsingular action of a locally compact group on a standard measure space. Suppose that $G \curvearrowright (X,\mu)$ is essentially free and ergodic and let $M = L^\infty(X,\mu) \rtimes G$.*

(a) If G is nondiscrete, then M is not finite.

(b) M is semifinite if and only if there exists a measure ν in the same measure class as μ such that

$$g \cdot \nu = \delta_G(g)\nu$$

for all $g \in G$.

In particular, if G is a nondiscrete group and (X,μ) is such that $g \cdot \mu = \delta_G(g)\mu$, then M is of type I_∞ or II_∞. This holds in particular if G is unimodular and $G \curvearrowright (X,\mu)$ is measure preserving.

The continuous core

Typically, semifinite von Neumann algebras are easier to study than nonsemifinite ones. Connes and Takesaki succeeded in [CT77] to associate to every (not necessarily semifinite) von Neumann algebra a canonical semifinite one, called the *continuous core*, which still contains a lot of information about the original von Neumann algebra, while being easier to study. In this section, we discuss this construction. The presentation here is based on [Tak03a, Chapter XII].

Let M be a von Neumann algebra. Fix a faithful, normal, semifinite weight φ on M. The modular automorphism group defines a von Neumann dynamical system $(\mathbb{R}, M, \sigma^\varphi)$, allowing the construction of a crossed product.

Definition 2.4.43. Let M be a von Neumann algebra and φ a faitful, normal, semifinite weight on M. The *continuous core* $c_\varphi(M)$ of M with respect to φ is the crossed product $M \rtimes_{\sigma^\varphi} \mathbb{R}$.

We denote by $L_\varphi(\mathbb{R})$ the canonical subalgebra $L(\mathbb{R}) \subseteq c_\varphi(M)$.

Using Connes' cocycle derivative theorem it is not hard to prove that for any two faithful, normal tracial weights φ and ψ, we have $c_\varphi(M) \cong c_\psi(M)$. Because of this, we will usually denote the continuous core by $c(M)$ and only write $c_\varphi(M)$ if we need the concrete realization of the continuous core for the weight φ.

Using (2.4.4) and the fact that $\varphi \circ \sigma_t^\varphi$ for $t \in \mathbb{R}$, one checks that the modular automorphism group for the dual weight $\tilde{\varphi}$ on $c_\varphi(M)$ is given by $\sigma_t^\varphi = \mathrm{Ad}(u_t)$ and hence that $c_\varphi(M)$ is semifinite. Denote the trace by Tr_φ.

Viewing $\mathbb{R}_0^+$ as the dual group of $\mathbb{R}$, there exists a so-called *dual action* $\mathbb{R}_0^+ \curvearrowright^{\theta^\varphi} c_\varphi(M)$ satisfying

$$\theta_s(x) = x \qquad \text{and} \qquad \theta_s(u_t) = s^{it} u_t$$

for $s \in \mathbb{R}_0^+$, $t \in \mathbb{R}$ and $x \in M$ (see [Tak03a, Theorem XII.1.1]). This action satisfies

$$\mathrm{Tr}_\varphi \circ \theta_s = s \ \mathrm{Tr}_\varphi .$$

Moreover, $M \mathbin{\overline{\otimes}} B\big(L^2(\mathbb{R})\big) \cong c_\varphi(M) \rtimes_\theta \mathbb{R}_0^+$. (In particular, $M \cong c_\varphi(M) \rtimes \mathbb{R}_0^+$ if M is a factor of type II_∞ or type III.)

If M is a factor, then θ^φ induces an *ergodic* action $\mathbb{R}_0^+ \curvearrowright^{\theta_\varphi} \mathcal{Z}\big(c_\varphi(M)\big)$ called *the flow of weights* (see [Tak03a, Corollary 1.4]). Using that $\mathcal{Z}\big(c_\varphi(M)\big)$ is abelian, we identify this flow with the induced action of $\mathbb{R}_0^+$ on a standard probability space (X, μ). The remarkable discovery of this flow is due to Takesaki in [Tak73].

The factor M is of type I or II if and only if this action is isomorphic to $\mathbb{R}_0^+ \curvearrowright \mathbb{R}_0^+$. If M is of type III, this action can be used to define the following finer classification. This classification is originally due to Connes in [Con73]. Its link with the flow of weights as presented here is due to Connes and Takesaki in [CT77].

Definition 2.4.44. Let M be a type III factor. Then, we are in one of the following cases.

(a) We say that M is of type *type III*$_1$ if the flow of weights is isomorphic to the action $\mathbb{R}_0^+ \curvearrowright \{x\}$ on a single point.

(b) We say that M is of *type III*$_\lambda$ if the flow of weights is isomorphic to the action $\mathbb{R}_0^+ \curvearrowright \mathbb{R}_0^+/\lambda^{\mathbb{Z}}$ with $0 < \lambda < 1$.

(c) We say that M is of *type III*$_0$ if the flow of weights is *properly ergodic*, i.e. every orbit has measure zero.

2.4.6 Tensor products

Just as in the case of the group von Neumann algebra and the crossed product, there is only one notion of tensor products of von Neumann algebras that makes sense.

Definition 2.4.45. Let $M \subseteq B(\mathcal{H})$ and $N \subseteq B(\mathcal{K})$ be two von Neumann algebras. The *(von Neumann) tensor product* $M \mathbin{\overline{\otimes}} N$ is the von Neumann algebra on $B(\mathcal{H} \otimes \mathcal{K})$ generated by $M \otimes_{\mathrm{alg}} N$, i.e.

$$M \mathbin{\overline{\otimes}} N = \overline{M \otimes_{\mathrm{alg}} N}^{\mathrm{w.o.}}$$

One can prove that the tensor product defined above is independent of the chosen representation of the von Neumann algebras (see for instance [Bla06, Theorem III.1.5.4]).

We will use the following example multiple times throughout this thesis.

Example 2.4.46. Let (X, μ) and (Y, ν) be two standard probability spaces. Then,

$$L^\infty(X, \mu) \otimes L^\infty(Y, \nu) \cong L^\infty(X \times Y, \mu \otimes \nu).$$

More generally, if M is a von Neumann algebra on $B(\mathcal{H})$, then the tensor product $L^\infty(X, \mu) \mathbin{\overline{\otimes}} M$ can be viewed as the von Neumann algebra of bounded measurable M-valued functions on X, acting on $L^\infty(X, \mu; \mathcal{H})$ by multiplication. This fact can be found in [Tak79, Section IV.7] and [Bla06, p. III.1.5.6].

A proof for the following easy result can be found in [Bla06, Proposition III.1.5.10]

Proposition 2.4.47. *Let M and N be von Neumann algebras. Then, $M \mathbin{\overline{\otimes}} N$ is a factor if and only if both M and N are factors.*

The following is a surprisingly deep theorem. The first proof of this theorem was one of the first applications of Tomita-Takesaki modular theory. Nowadays, there are also more elementary proofs (see for instance [Tak79, Theorem IV.5.9]).

Theorem 2.4.48 (Commutation theorem for tensor products)**.** *Let $M \subseteq B(\mathcal{H})$ and $N \subseteq B(\mathcal{K})$ be two von Neumann algebras. Then,*

$$(M \mathbin{\overline{\otimes}} N)' = M' \mathbin{\overline{\otimes}} N'.$$

A consequence of the previous theorem is the following (see [Bla06, Corollary III.4.5.9]).

Corollary 2.4.49. *Let $M_i \subseteq B(\mathcal{H}_i)$ be von Neumann algebras for $i = 1, 2$. Then,*

$$(M_1 \mathbin{\overline{\otimes}} M_2) \cap (N_1 \mathbin{\overline{\otimes}} N_2) = (M_1 \cap N_1) \mathbin{\overline{\otimes}} (M_2 \cap N_2).$$

Similar to completely bounded maps on minimal tensor products of C*-algebras, *normal* completely bounded maps on von Neumann algebras induce normal completely bounded maps on tensor products. A proof can be found in [Tak79, Proposition IV.5.13].

Theorem 2.4.50. *Let M_1, M_2, N_1 and N_2 be von Neumann algebras. Given normal c.b. maps $\Phi_i : M_1 \to N_i$, there exists a unique normal c.b. map*

$$\Phi_1 \otimes \Phi_2 : M_1 \mathbin{\overline{\otimes}} M_2 \to N_1 \mathbin{\overline{\otimes}} N_2$$

satisfying $(\Phi_1 \otimes \Phi_2)(x_1 \otimes x_2) = \Phi_1(x_1) \otimes \Phi_2(x_2)$ for $x_i \in M_i$. Moreover, $\|\Phi_1 \otimes \Phi_2\|_{cb} = \|\Phi_1\|_{cb} \|\Phi_2\|_{cb}$

For normal linear functionals, the previous yields the following result (which can also be proven directly by writing ϕ and ψ as vector states).

Corollary 2.4.51. *Let M and N be two von Neumann algebras. Let $\varphi : M \to \mathbb{C}$ and $\psi : N \to \mathbb{C}$ be two normal linear functionals. Then, there exists a normal linear functional*

$$\varphi \otimes \psi : M \mathbin{\overline{\otimes}} N \to \mathbb{C}$$

satisfying $(\varphi \otimes \psi)(x \otimes y) = \varphi(x)\psi(y)$ for $x \in M$ and $y \in N$. Moreover, $\|\varphi \otimes \psi\| = \|\varphi\| \|\psi\|$. If φ and ψ are tracial, then also $\varphi \otimes \psi$ is tracial.

2.4.7 Modules over von Neumann algebras

Modules, and in particular bimodules, are an important tool in the study of von Neumann algebras. The notion of bimodules (originally called *correspondences*) is originally due to Connes in the 1980, motivated by his work on property (T) for von Neumann algebras [CJ85]. His original notes were never published, but his motivation and the original applications can be found in his book [Con94, Appendix V.B]. The results in this section are based on [AP14, Chapter 12] and [Tak03a, Section IX.3].

Definition 2.4.52. Let M and N be von Neumann algebras.

(a) A *left M-module* is a Hilbert space $\mathcal{H}$ equipped with a normal, unital $*$-morphism $\pi_\ell : M \to B(\mathcal{H})$.

(b) A *right M-module* is a Hilbert space $\mathcal{H}$ equipped with a normal, unital $*$-morphism $\pi_r : M^{\mathrm{op}} \to B(\mathcal{H})$.

(c) An *M-N bimodule* is a Hilbert space $\mathcal{H}$ that is both a left M-module and a right N-module such that the representations $\pi_\ell : M \to B(\mathcal{H})$ and $\pi_r : N^{\mathrm{op}} \to B(\mathcal{H})$ commute.

Here, M^{op} denotes the algebra M with the same addition, but opposite multiplication, i.e. $x^{\mathrm{op}}y^{\mathrm{op}} = (yx)^{\mathrm{op}}$. When $\mathcal{H}$ is a left M-module, a right N-module or an M-N-bimodule, we denote the resulting operations by $x\xi = \pi_\ell(x)\xi$, $\xi y = \pi_r(y)\xi$ and $x\xi y = \pi_\ell(x)\pi_r(y)\xi$ respectively. Note that the commutativity for the representations π_ℓ and π_r in the definition of a bimodule precisely means that $x(\xi y) = (x\xi)y$, so that the notation $x\xi y$ is well defined.

The easiest example of an M-M bimodule is the module given by the GNS-representation. Indeed, if φ is a faithful, normal, semifinite weight on M, then the GNS-Hilbert space $L^2(M, \varphi)$ is naturally a bimodule for the actions $x\xi y = xJy^*J\xi$, where J denotes the modular conjugation. This bimodule is called the *trivial bimodule.*

Another example is the following. If M and N are two von Neumann algebras with faithful, normal, semifinite weights φ and ψ respectively, then the Hilbert space $L^2(M, \varphi) \otimes L^2(N, \psi)$ is an M-N-bimodule for the operations

$$x(\xi \otimes \eta)y = x\xi \otimes Jy^*J\eta,$$

where again J denotes the modular conjugation. This module is called the *coarse M-N-bimodule.*

If $\mathcal{H}$ and $\mathcal{K}$ are two M-N-bimodules, then we say that $\mathcal{H}$ is *isomorphic* or *unitarily equivalent* to $\mathcal{K}$ if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ that intertwines both the left M- and the right N-operation, i.e. $x(U\xi)y = U(x\xi y)$ for all $\xi \in \mathcal{H}$, all $x \in M$ and all $y \in N$. We say that $\mathcal{H}$ is *contained* in $\mathcal{K}$ and write $\mathcal{H} \leqslant \mathcal{K}$ if $\mathcal{H}$ is isomorphic to a closed subspace $\mathcal{H}' \subseteq \mathcal{K}$ that is satisfying $x\mathcal{H}'y \subseteq \mathcal{H}'$ for all $x \in M$ and $y \in N$.

There is also a weaker notion of containment for bimodules of von Neumann algebras. Readers familiar to group representations will see that this notion is an analogue of weak containment for group representations. The definition of weak containment and the related notion of the Fell topology on the level of bimodules is due to Connes and Jones in [CJ85]. The definition given here can be found in [Ana95]. Given an M-N-bimodule $\mathcal{H}$ and $\xi \in \mathcal{H}$, we call the map

$$M \otimes_{\text{alg}} N \to \mathbb{C} : x \otimes y \mapsto \langle x\xi y, \xi \rangle$$

a *positive coefficient* of the bimodule $\mathcal{H}$.

Definition 2.4.53. Let M and N be two tracial von Neumann algebras. We say that an M-N-bimodule $\mathcal{H}$ is *weakly contained* in an M-N-bimodule $\mathcal{K}$ and write $\mathcal{H} \prec \mathcal{K}$ if the positive coefficients of $\mathcal{H}$ can be approximated by the finite sums of positive coefficients of $\mathcal{K}$, i.e. if for all $\xi \in \mathcal{H}$, all $\varepsilon > 0$ and all finite subsets $E \subseteq M$ and $F \subseteq N$, there exist $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that

$$\left| \langle x\xi y, \xi \rangle - \sum_{i=1}^{n} \langle x\eta_i y, \eta_i \rangle \right| < \varepsilon$$

for all $x \in E$ and $y \in F$.

Every M-N-bimodule $\mathcal{H}$ gives rise to a $*$-representation $\pi_{\mathcal{H}} : M \otimes_{\text{alg}} N \to B(\mathcal{H})$. Weak containment can also be expressed in terms of the norms of these $*$-representations. See [Ana95] for details.

Theorem 2.4.54. *Let M and N be two von Neumann algebras. Suppose that $\mathcal{H}$ and $\mathcal{K}$ are two M-N bimodules. Then, the following are equivalent*

- $\mathcal{H} \prec \mathcal{K}$,
- $\|\pi_{\mathcal{H}}(x)\| \leqslant \|\pi_{\mathcal{K}}(x)\|$ *for all* $x \in M \otimes_{alg} N$.

One can easily prove that an M-M-bimodule $\mathcal{H}$ of tracial von Neumann algebras M contains the trivial bimodule $L^2(M)$ if and only if $\mathcal{H}$ admits an M-central and tracial vector, i.e. a vector $\xi \in \mathcal{H}$ such that $x\xi = \xi x$ and $\langle x\xi, \xi \rangle = \tau(x)$ for all $x \in M$. The following theorem states that *weak* containment of the trivial bimodule is characterized by the existence of *almost* M-central vectors that are *almost* tracial. A proof can be found in [AP14, Proposition 12.3.11].

Proposition 2.4.55. *Let (M, τ) be a tracial von Neumann algebra, $N \subseteq M$ a von Neumann subalgebra and $\mathcal{H}$ an M-N-bimodule. Then, the trivial M-N-bimodule $L^2(M)$ is weakly contained in $\mathcal{H}$ if and only if there exists a net of vectors $(\xi_i)_i$ in $\mathcal{H}$ such that*

$$\lim_i \|x\xi_i - \xi_i x\| = 0 \qquad \textit{and} \qquad \lim_i \tau(x) = \langle x\xi_i, \xi_i\rangle$$

for all $x \in N$.

Right bounded vectors and Connes' tensor product

Let N be a von Neumann algebra with faithful, semifinite, normal weight φ and $\mathcal{H}$ a right N-module. Denote by $L^2(N, \varphi)$ the GNS-Hilbert space. As mentioned before, this subspace is naturally a right N-module for the module operation defined by $\xi x = Jx^*J\xi$ for $x \in N$ and $\xi \in L^2(N, \varphi)$. Writing $\eta'_\varphi(x) = J\eta_\varphi(x^*)$ for $x \in \mathfrak{n}_\varphi^* = \{x \in N \mid \varphi(xx^*) < +\infty\}$, we have

$$\eta'_\varphi(x)y = \eta'_\varphi(xy)$$

for all $x \in \mathfrak{n}_\varphi^*$ and $y \in N$.

For every $\xi \in \mathcal{H}$, the (possibly unbounded) operator

$$\eta'_\varphi(\mathfrak{n}_\varphi^*) \to \mathcal{H} : \eta'_\varphi(x) \mapsto \xi x$$

is densely defined operator $L^2(N, \varphi) \to \mathcal{H}$. We denote this operator by L_ξ.

Definition 2.4.56. Let N be a von Neumann algebra and $\mathcal{H}$ a right N-module. A vector $\xi \in \mathcal{H}$ is called *right bounded* if L_ξ is a bounded operator, i.e. if there exists a $C > 0$ such that $\|\xi x\| \leqslant C \|\eta_\varphi(x)\|$ for any $x \in \mathfrak{n}_\varphi^*$.

We denote by $\mathcal{H}^0$ the space of right bounded vectors. This space is dense in $\mathcal{H}$ (see [Tak03a, Lemma IX.3.3]). The map $\xi \mapsto L_\xi$ gives an identification of $\mathcal{H}^0$ with a subspace of $B\big(L^2(N, \varphi)_N, \mathcal{H}_N\big)$ of bounded operators $L^2(N, \varphi) \to \mathcal{H}$ that commute with the right N-module operation. If φ is a state, the image of this map is the whole of $B\big(L^2(N, \varphi)_N, \mathcal{H}_N\big)$.

Given $\xi, \eta \in \mathcal{H}^0$, the operator $L_\xi^* L_\eta \in B\big(L^2(N, \varphi)\big)$ commutes with the right N-module operation on $L^2(N, \varphi)$ and hence $L_\xi^* L_\eta \in N$. This allows the definition of the following N-valued inner product on $\mathcal{H}^0$ by

$$\langle \xi, \eta \rangle_M = L_\eta^* L_\xi$$

for $\xi, \eta \in \mathcal{H}^0$.

The following construction is due to Connes. If $\mathcal{K}$ is a left N-module, then we define a positive, symmetric sesquilinear form on $\mathcal{H}^0 \otimes_{\text{alg}} \mathcal{H}$ by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \langle \xi_1, \xi_2 \rangle_M \eta_1, \eta_2 \rangle = \left\langle L_{\xi_2}^* L_{\xi_1} \eta_1, \eta_2 \right\rangle$$

for $\xi_1, \xi_2 \in \mathcal{H}$ and $\eta_1, \eta_2 \in \mathcal{K}$. By separation and completion, we obtain a Hilbert space $\mathcal{H} \otimes_N \mathcal{K}$. We will denote the projection of an elementary tensor $\xi \otimes \eta$ for $\xi \in \mathcal{H}^0$ and $\eta \in \mathcal{K}$ by $\xi \otimes_N \eta$. Note that

$$\xi x \otimes_N \eta = \xi \otimes_N x\eta$$

for $\xi \in \mathcal{H}^0$, $\eta \in \mathcal{K}$ and $x \in N$.

If $\mathcal{H}$ is an M-N-bimodule and $\mathcal{K}$ is an N-P-bimodule, then $\mathcal{H} \otimes_N \mathcal{K}$ is an M-P-bimodule for the operations

$$x(\xi \otimes_N \eta)y = x\xi \otimes_N \eta y.$$

This bimodule is called the *(Connes) tensor product* or *composition* of $\mathcal{H}$ and $\mathcal{K}$. More details on this construction can be found in [Tak03a, Corollary IX.3.8].

2.4.8 Amenability and relative amenability

In this section, we introduce a notion of amenability for von Neumann algebras. This notion was first introduced by J. Schwartz [Sch63] under the name Property (P). The definition used here was proven to be equivalent to this notion of property (P) by Hakeda and Tomiyama in [HT67]. More details on the notions in this section can be found in [Bla06, Section IV.2] and [Tak03b, Chapter XVI].

Definition 2.4.57. A von Neumann algebra $M \subseteq B(\mathcal{H})$ is said to be *amenable* or *injective* if there exists a (not necessarily normal) conditional expectation $E : B(\mathcal{H}) \to M$.

This notion is independent from the chosen representation by the following proposition. A proof can be found in [AP14, Proposition 10.2.2].

Proposition 2.4.58. *A von Neumann algebra M is injective if and only if for every inclusion $A \subseteq B$ of unital C^*-algebras, every u.c.p. map $\Phi : A \to M$ extends to a u.c.p. map $B \to M$.*

Obviously, $B(\mathcal{H})$ is an amenable von Neumann algebra for every Hilbert space $\mathcal{H}$. Also abelian von Neumann algebras are amenable (see for instance [Bla06, Corollary IV.2.2.10]).

For a countable group Γ, the group von Neumann algebra $L(\Gamma)$ is amenable if and only if Γ is amenable (see for instance [AP14, Theorem 10.1.3]). Also, the group measure space construction $L^\infty(X) \rtimes \Gamma$ for a pmp action $\Gamma \curvearrowright (X, \mu)$ of a discrete group is amenable if and only if Γ is amenable.

When G is not countable, the situation is once again more complicated. Similarly to the situation of the reduced crossed product, $L(G)$ can be amenable even though G is not amenable. Indeed, for instance the group von Neumann algebra of all connected groups is amenable (see [Con76, Theorem 6.9]). However, Lau and A. L. T. Paterson proved the following result in [LP91, Corollary 3.2].

Theorem 2.4.59. *Let G be a locally compact group G, then the following are equivalent.*

(i) *G is amenable,*

(ii) *G is inner amenable and $L(G)$ is amenable.*

Just as for amenability of groups, there exist many different characterizations of amenability for von Neumann algebras. We mention the following characterization for tracial von Neumann algebras. If $N \subseteq M$ is an inclusion of von Neumann algebras, then we say that a state $\varphi : M \to \mathbb{C}$ is N*-central* if $\varphi(xy) = \varphi(yx)$ for every $x \in N$ and $y \in M$. A proof can be found in [AP14, Proposition 10.2.5].

Theorem 2.4.60. *A tracial von Neumann algebra (M, τ) is amenable if and only if there exists an M-central state $\Omega : B\big(L^2(M)\big) \to \mathbb{C}$ such that $\Omega|_M = \tau$.*

The following famous theorem of Connes from [Con76] establishes that all amenable von Neumann algebras are approximately finite dimensional.

Theorem 2.4.61. *A von Neumann algebra M is amenable if and only if it is* approximately finite dimensional *or* hyperfinite*, i.e. there exists an increasing sequence $(M_n)_n$ of finite-dimensional von Neumann subalgebras such that $\bigcup_n M_n$ is w.o. dense in M.*

The approximately finite dimensional factors (and hence the amenable factors) are completely classified by their type and (in the type III case) by their flow of weights. The II_1 case was already proven by Murray and von Neumann in [MvN43]. The type III case was only obtained much later. In [Con75a; Con75b; Con76] Connes was able to prove the uniqueness of the AFD factors of type II_∞ and type III_λ for $0 \leqslant \lambda < 1$. In [Haa87] Haagerup was finally able to settle the type III_1 case.

Theorem 2.4.62. *We have the following.*

(a) For every type I_n $(n = 1, 2, \ldots, \infty)$, II_1, II_∞, III_λ $(0 < \lambda \leqslant 1)$, there exists exactly one approximately finite dimensional factor of that type.

(b) For every properly ergodic action $\mathbb{R}_0^+ \curvearrowright (X, \mu)$, there exists exactly one hyperfinite factor such that its flow of weights is isomorphic to this flow.

In particular, all group von Neumann algebras of all countable, amenable, icc groups and all group measure space constructions for free, ergodic, pmp actions of countable groups are isomorphic.

Relative amenability

A relative version of amenability for tracial von Neumann algebras was introduced by Popa in [Pop86]. More details can be found in [OP10b, Section 2.2] and [AP14, Paragraph 12.4.3].

Let (M, τ) be a tracial von Neumann algebra and $B \subseteq M$ a subalgebra. The *Jones basic construction* for $B \subseteq M$ is the von Neumann algebra $\langle M, e_B \rangle \subseteq B\big(L^2(M)\big)$ generated by M and the orthogonal projection e_B from $L^2(M)$ onto the subspace $L^2(B)$. Jones [Jon83] proved that $\langle M, e_B \rangle = B\big(L^2(M)\big) \cap (JBJ)'$, where J is the modular conjugation of M (see for instance [AP14, Proposition 9.4.2]). Moreover, $\langle M, e_B \rangle$ is a semifinite von Neumann algebra (see [AP14, Section 9.4]).

Popa introduced the following notion and proved the equivalent characterizations below in [Pop86, Theorem 3.2.3]. A proof can also be found in [OP10b, Therem 2.1].

Theorem 2.4.63. *Let (M, τ) be a tracial von Neumann algebra and $B \subseteq M$ a von Neumann subalgebra. Then, the following are equivalent.*

(i) There exists a conditional expectation $\Phi : \langle M, e_B \rangle \to M$.

(ii) There exists an M-central state $\Omega : \langle M, e_B \rangle \to \mathbb{C}$ such that $\Omega|_M = \tau$.

We say that M is amenable relative to B *if one the these conditions is satisfied.*

Note that M is amenable relative to $\mathbb{C}$ if and only if M is amenable. By [OP10b, Theorem 2.4 (3)], it follows that M is amenable relative to an amenable subalgebra if and only if M itself is amenable. If (Γ, N, α) is a von Neumann dynamical system with Γ a countable group, N a tracial von Neumann algebra and $\Gamma \curvearrowright^{\alpha} N$ trace-preserving, then the crossed product $M = N \rtimes \Gamma$ is amenable relative to N if and only if Γ is amenable (see [Ana79, Proposition 4.1] or [Pop86, Theorem 3.2.4 (3)]).

Amenability for bimodules

In [PV14a, Definition 2.2], Popa and Vaes introduced an even more general notion of amenability on bimodules.

Theorem 2.4.64. *Let (M, τ) and (Q, τ) be tracial von Neumann algebras and $P \subseteq M$ be a von Neumann subalgebra. Suppose that $\mathcal{H}$ is an M-Q-bimodule. Denote by $\pi_r : Q^{op} \to B(\mathcal{H})$ the representation from the right Q-operation and set $\mathcal{M} = B(\mathcal{H}) \cap \pi_r(Q^{op})'$. Then, the following are equivalent.*

- *(i) There exists a P-central state $\Omega : \mathcal{M} \to \mathbb{C}$ such that $\Omega|_M = \tau$.*
- *(ii) There exists a conditional expectation $\Phi : \mathcal{M} \to P$ such that $\Phi|_M = E_P$, where $E_P : M \to P$ denotes the unique trace preserving conditional expectation.*

If one of the above conditions hold, then we say that $\mathcal{H}$ is left P-amenable*.*

Note that M is amenable relative to a subalgebra B if and only if the M-B-module $L^2(M)$ is left M-amenable.

2.4.9 Solidity, strong solidity and stable strong solidity

In this section, we discuss several versions of solidity properties. Solidity properties are indecomposability properties for von Neumann algebras. The first such property was introduced by Ozawa in [Oza04] for finite von Neumann algebras. The version for general von Neumann algebras was introduced by Vaes and Vergnioux in [VV07, Definition 2.2]. Recall that a von Neumann algebra is called diffuse if it does not contain any minimal projections.

Definition 2.4.65. A von Neumann algebra M is called *solid* if for any diffuse von Neumann subalgebra $A \subseteq M$ with expectation, its relative commutant $A' \cap M$ is amenable.

As mentioned in the introduction, it was proved in [Oza04, Theorem 1] that the group von Neumann algebra of all countable, discrete groups in class $\mathcal{S}$ are solid. Solidity for nonamenable factors implies primeness in the following sense.

Definition 2.4.66. A factor M is called prime if it can not be decomposed as a tensor product $M \cong M_1 \mathbin{\overline{\otimes}} M_2$ for factors M_1 and M_2 not of type I.

The notion of primeness was originally introduced by Popa in [Pop83], where he proved that the group von Neumann algebra $L(\mathbb{F}_S)$ of the free group $\mathbb{F}_S$ on an uncountable number of generators is prime. The same result for group von Neumann algebras $L(\mathbb{F}_n)$ of the free groups $\mathbb{F}_n$ on finite and countable number of generators was first proven by Ge in [Ge98].

A stronger version of solidity was introduced by Ozawa and Popa in [OP10b]. Recall that given a von Neumann subalgebra $A \subseteq M$, its *normalizer* is $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$.

Definition 2.4.67. A von Neumann algebra M is called *strongly solid* if for every diffuse, amenable von Neumann subalgebra $A \subseteq M$ with expectation, we have that the normalizer $\mathcal{N}_M(A)''$ remains amenable.

By [CS78, Corollary 8] every diffuse von Neumann subalgebra with expectation contains an amenable (even abelian), diffuse subalgebra with expectation. Hence, as the name suggests, strong solidity implies solidity. Moreover, strong solidity in nonamenable von Neumann algebras also implies the absence of a Cartan subalgebra. In particular, nonamenable, strongly solid von Neumann algebras can not be decomposed as a group measure space von Neumann algebras.

In [OP10a, Corollary B], it was proven that the free group factors $L(\mathbb{F}_n)$ for $2 \leqslant n \leqslant +\infty$ are strongly solid. This provided a unified proof for Voiculescu's result [Voi96, Theorem 5.3] that $L(\mathbb{F}_n)$ does not contain any Cartan subalgebra and Ozawa's result in [Oza04] that $L(\mathbb{F}_n)$ is solid.

Motivated by the fact that strong solidity is not preserved under infinite amplifications (i.e. taking a tensor product with $B(\mathcal{H})$), Boutonnet, Houdayer, and Vaes introduced the following even stronger property. Recall that the *stable normalizer* $\mathcal{N}^{\mathrm{s}}_M(A)$ of a subalgebra $A \subseteq M$ is given by

$$\mathcal{N}^{\mathrm{s}}_M(A) = \{x \in M \mid xAx^* \subseteq A \text{ and } x^*Ax \subseteq A\}$$

Definition 2.4.68. A von Neumann algebra M is called *stably strongly solid* if for every diffuse, amenable von Neumann subalgebra $A \subseteq M$ with expectation, we have that the stable normalizer $\mathcal{N}^{\mathrm{s}}_M(A)''$ remains amenable.

It was proven in [BHV18, Theorem 3.8] that, among others, the free group factors $L(\mathbb{F}_n)$ for $2 \leqslant n \leqslant +\infty$ are stable strongly solid.

The following result is [BHV18, Lemma 3.4].

Proposition 2.4.69. *Let M be a von Neumann algebra and $A \subseteq M$ a von Neumann subalgebra.*

(a) For every projection $p \in A$, we have $p\mathcal{N}_M^s(A)p = \mathcal{N}_{pMp}^s(pAp)$.

(b) For every infinite dimensional Hilbert space, we have

$$\mathcal{N}_M^s(A)'' \mathbin{\overline{\otimes}} B(\mathcal{H}) = \mathcal{N}^s_{M \mathbin{\overline{\otimes}} B(\mathcal{H})}\big(A \mathbin{\overline{\otimes}} B(\mathcal{H})\big)'' = \mathcal{N}_{M \mathbin{\overline{\otimes}} B(\mathcal{H})}\big(A \mathbin{\overline{\otimes}} B(\mathcal{H})\big)''.$$

Using the previous, one proves that stable strong solidity is stable under amplifications. See [BHV18, Corollary 5.2] for a detailed proof.

Theorem 2.4.70. *Any diffuse von Neumann algebra M is stably strongly solid if and only if its infinite amplification $M \mathbin{\overline{\otimes}} B(\mathcal{H})$ is strongly solid, where $\mathcal{H}$ is any infinite dimensional Hilbert space.*

2.5 Countable equivalence relations

Motivated by the work of Dye [Dye59; Dye63] and Krieger [Kri69a; Kri69b] on orbit equivalence relations of group actions, Feldman and Moore introduced the notion of a countable, Borel equivalence relation. We will discuss parts of the theory about countable, Borel equivalence relations below. As we will see, many notions for these relations have an analogue in the theory of von Neumann algebras. We refer to [FM77a; FM77b] for more details.

Definition 2.5.1. Let X be a standard Borel space and $\mathcal{R}$ an equivalence relation on X. We say that

(a) the equivalence relation $\mathcal{R}$ is *countable* if all equivalence classes of $\mathcal{R}$ are countable,

(b) the equivalence relation $\mathcal{R}$ is *Borel* if $\mathcal{R} \subseteq X \times X$ is a Borel subset

The main examples of countable, Borel equivalence relations are the following. Let $\Gamma \curvearrowright X$ be an action by Borel automorphisms of a countable group Γ. Then, the *orbit equivalence relation* $\mathcal{R}(\Gamma \curvearrowright X)$ defined by

$$\mathcal{R}(\Gamma \curvearrowright X) = \{(gx, x) \in X \times X \mid g \in \Gamma, x \in X\}$$

is a countable, Borel equivalence relation. By [FM77a, Theorem 1] every countable equivalence relation $\mathcal{R}$ is actually the orbit equivalence relation of some group action. However, this action is not canonical and many notions and constructions only depend on the countable equivalence relation, and not on the underlying group action. As we will see in Section 2.5.3 the group measure space construction by countable groups is an example of such a construction. As a result, many constructions are more natural when stated in terms of countable equivalence relations instead of in terms of group actions.

Let $\mathcal{R}$ be a countable, Borel equivalence relation. For $x, y \in X$, we will write $x \sim_{\mathcal{R}} y$ or $x \sim y$ if $(x, y) \in \mathcal{R}$. We denote by p_1 and p_2 the projections from $\mathcal{R}$ to X on the first and second coordinate respectively. These maps are bimeasurable by [Kec95, Theorem 18.10]. Given a measurable set $E \subseteq X$, we define the *$\mathcal{R}$-saturation* of E as $[E]_{\mathcal{R}} = \{x \in X \mid \exists y \in E : x \sim y\}$. For any Borel subset $E \subseteq X$, we denote by $\mathcal{R}|_E = \mathcal{R} \cap (E \times E)$ the restricted equivalence relation. We denote $\mathcal{R}^{(2)} = \{(x, y, z) \in X \times X \times X \mid x \sim y \sim z\}$. Note that $\mathcal{R}^{(2)}$ is also a Borel subset.

When the measurable space X is endowed with a (σ-finite) measure μ, one defines the following.

Definition 2.5.2. Let (X, μ) be a standard measure space and $\mathcal{R}$ a countable, Borel equivalence relation on X. The relation $\mathcal{R}$ is said to be *nonsingular* for μ if for all measurable $E \subseteq X$ with $\mu(E) = 0$, we have $\mu\big([E]_{\mathcal{R}}\big) = 0$.

It is clear that nonsingularity only depends on the measure class of μ in the sense that if μ' is another measure in the same measure class of μ (meaning that μ and μ' have the same sets of measure zero), then $\mathcal{R}$ is nonsingular for μ' if and only if it is nonsingular for μ. It is also clear that an orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ is nonsingular if and only if $\Gamma \curvearrowright (X, \mu)$ is nonsingular.

In what follows, we will always assume all countable equivalence relations to be Borel and nonsingular.

Given a set $\mathcal{W} \subseteq \mathcal{R}$, we denote its sections by

$$ {}_x\mathcal{W} = \{y \in X \mid (x, y) \in \mathcal{W}\} \qquad \text{and} \qquad \mathcal{W}_y = \{x \in X \mid (x, y) \in \mathcal{W}\} $$

for $x, y \in X$. We define the following notions for sets with 'small' sections.

Definition 2.5.3. Let $\mathcal{R}$ be a countable equivalence relation on a standard measure space (X, μ). A Borel subset $\mathcal{W} \subseteq \mathcal{R}$ is called *bounded* if the number of elements in its sections is bounded, i.e. if there exists a $C > 0$ such that $|{}_x\mathcal{W}| < C$ and $|\mathcal{W}_y| < C$ for a.e. $x, y \in X$.

We call a Borel subset $\mathcal{W} \subseteq \mathcal{R}$ *locally bounded* if for every $\varepsilon > 0$ there exists a Borel subset $E \subseteq X$ with $\mu(X \setminus E) < \varepsilon$ such that $\mathcal{W} \cap (E \times E)$ is bounded.

To any countable equivalence relation, one associates the following group and pseudogroup.

Definition 2.5.4. Let $\mathcal{R}$ be a countable equivalence relation on a standard measure space (X, μ). The *full group* $[\mathcal{R}]$ is the group of all Borel automorphisms $\varphi : X \to X$, identified up to almost everywhere equality, for which $\operatorname{graph}\varphi = \{(\varphi(x), x)\}_{x\in X}$ is contained in $\mathcal{R}$.

The *full pseudogroup* $[[\mathcal{R}]]$ is the set of all partial Borel automorphisms $\varphi : A \to B$ for Borel sets $A, B \subseteq X$ for which $\operatorname{graph}\varphi$ is contained in $\mathcal{R}$. Again, these partial automorphisms are identified up to almost everywhere equality.

Using [Kec95, Theorem 18.10], one checks that $\mathcal{R}$ can be written as a countable union $\mathcal{R} = \bigcup_i \operatorname{graph}\varphi_i$ of elements $\varphi_i \in [\mathcal{R}]$. Moreover, if we allow φ_i to be in $[[\mathcal{R}]]$, then we can assume the union to be disjoint. Every bounded Borel subset $\mathcal{W} \subseteq \mathcal{R}$ can be written as a finite union of graphs of elements in $[[\mathcal{R}]]$.

An important class of countable equivalence relations is the following.

Definition 2.5.5. A countable equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is called *ergodic* if for any measurable $E \subseteq \mathcal{R}$ with $E = [E]_{\mathcal{R}}$ up to null sets, we have $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

Ergodic countable equivalence relations play the same role in the theory of countable equivalence relations as factors do in the theory of von Neumann algebras. Indeed, every countable equivalence relation can be decomposed into ergodic ones as explained in [FM77a, p. 297]. Moreover, we will see that under the construction in Section 2.5.3 ergodic countable equivalence relations are precisely the ones that give rise to factors.

The notion of isomorphism for countable equivalence relations is the following.

Definition 2.5.6. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be countable equivalence relations on (X_1, μ_1) and (X_2, μ_2) respectively. A Borel isomorphism $\theta : X_1 \to X_2$ is called an *isomorphism of countable equivalence relations* if $\theta([x]_{\mathcal{R}_1}) = [\theta(x)]_{\mathcal{R}_2}$ for a.e. $x \in X_1$ and if $\theta_*\mu_1$ and μ_2 belong to the same measure class.

The condition $\theta([x]_{\mathcal{R}_1}) = [\theta(x)]_{\mathcal{R}_2}$ for a.e. $x \in X_1$ is equivalent to $(\theta \times \theta)(\mathcal{R}_1) = \mathcal{R}_2$ up to null sets when we equip $X \times X$ with the measure (counting measure on X) $\otimes\, \mu$ and $Y \times Y$ with (counting measure on Y) $\otimes\, \nu$. Obviously, if θ is an isomorphism of countable equivalence relations then $\varphi \mapsto \theta \circ \varphi \circ \theta^{-1}$ is an isomorphism of the associated full (pseudo-)groups.

Two nonsingular actions $\Gamma_1 \curvearrowright (X_1, \mu_1)$ and $\Gamma_2 \curvearrowright (X_2, \mu_2)$ are called *orbit equivalent* if their orbit equivalence relations $\mathcal{R}(\Gamma_1 \curvearrowright X_1)$ and $\mathcal{R}(\Gamma_2 \curvearrowright X_2)$ are isomorphic. An isomorphism between these orbit equivalence relations is called an *orbit equivalence*.

A weaker notion of isomorphism is the following. If $\mathcal{R}$ is a countable equivalence relation on (X, μ) and $E \subseteq X$ is a Borel subset, then we denote by $\mathcal{R}|_E = \mathcal{R} \cap (E \times E)$ its *restriction* to E.

Definition 2.5.7. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be countable equivalence relations on (X_1, μ_1) and (X_2, μ_2) respectively. We say that $\mathcal{R}_1$ is *stably isomorphic* to $\mathcal{R}_2$ if there exist measurable subsets $E_i \subseteq X_i$ of positive measure such that the restrictions $\mathcal{R}_1|_{E_1}$ and $\mathcal{R}_2|_{E_2}$ are isomorphic.

If $\mathcal{R}_1$ and $\mathcal{R}_2$ are stably isomorphic, then we write $\mathcal{R}_1 \cong_s \mathcal{R}_2$.

2.5.1 The counting measure

One defines two natural σ-finite measures on $\mathcal{R}$.

Definition 2.5.8. Let $\mathcal{R}$ be a countable equivalence relation on a standard measure space (X, μ). We define the *left (resp. right) counting measure* ν_ℓ (resp. ν_r) on $\mathcal{R}$ by

$$\nu_\ell(\mathcal{W}) = \int_X |{}_x\mathcal{W}| \, \mathrm{d}\mu(x) \qquad \text{and} \qquad \nu_r(\mathcal{W}) = \int_X |\mathcal{W}_y| \, \mathrm{d}\mu(y)$$

for any measurable set $\mathcal{W} \subseteq \mathcal{R}$. Here, as before, ${}_x\mathcal{W} = \{y \in X \mid (x, y) \in \mathcal{W}\}$ and $\mathcal{W}_y = \{x \in X \mid (x, y) \in \mathcal{W}\}$.

By [FM77a, Theorem 2], both the left and the right counting measures are indeed well-defined σ-finite measures. Note that ν_ℓ and ν_r are the restrictions of $\mu \otimes$(counting measure on X) and (counting measure on X)$\otimes \mu$ to $\mathcal{R}$ respectively.

The measures ν_ℓ and ν_r belong to the same measure class. The Radon-Nikodym derivative $D = \mathrm{d}\nu_\ell / \mathrm{d}\nu_r$ is called the *Radon-Nikodym cocycle* of $\mathcal{R}$ with respect to μ. For every partial Borel automorphism $\varphi : A \to B$ in $[[\mathcal{R}]]$ the measures $\varphi_* \mu|_A = \mu|_A \circ \varphi^{-1}$ and $\mu|_B$ belong to the same measure class and

$$\frac{\mathrm{d}\varphi_* \mu|_A}{\mathrm{d}\mu|_B}(x) = D\big(\varphi^{-1}(x), x\big) \tag{2.5.1}$$

for a.e. $x \in B$ (see [FM77a, Proposition 2.2]). In the same way, one also defines a natural measure class on $\mathcal{R}^{(2)}$.

As with ergodicity of actions, we can characterize ergodicity of $\mathcal{R}$ in terms of invariant functions.

Definition 2.5.9. Let $\mathcal{R}$ be a countable equivalence relation on (X, μ). A function $f : X \to \mathbb{C}$ is called $\mathcal{R}$-invariant if $f(x) = f(y)$ for a.e. $(x, y) \in \mathcal{R}$.

We denote by $L^p(X)^{\mathcal{R}}$ the $\mathcal{R}$-invariant functions in $L^p(X)$ for $1 \leqslant p \leqslant +\infty$.

We have the following.

Proposition 2.5.10. *Let $\mathcal{R}$ be a countable equivalence relation on (X, μ). Then, the following are equivalent.*

(i) *$\mathcal{R}$ is ergodic.*

(ii) $L^\infty(X)^{\mathcal{R}} = \mathbb{C}1$

(iii) *The only measurable invariant functions $f : X \to \mathbb{C}$ are almost everywhere constant.*

We will use the following definition in this thesis.

Definition 2.5.11. Let G be a locally compact group and $\mathcal{R}$ a countable equivalence relation. We call a Borel map $\omega : \mathcal{R} \to G$ a *cocycle* if

$$\omega(x, y)\omega(y, z) = \omega(x, z)$$

for a.e. $(x, y, z) \in \mathcal{R}^{(2)}$.

Note that for every cocycle $\omega : \mathcal{R} \to G$, we have $\omega(x, x) = e$ for a.e. $x \in X$.

As the name suggests, by [FM77a, Corollary 2], the Radon-Nikodym cocycle D above is indeed a cocycle $D : \mathcal{R} \to \mathbb{R}_0^+$. Given a free action $\Gamma \curvearrowright (X, \mu)$ of a countable group, its orbit equivalence relation $\mathcal{R}(G \curvearrowright X)$ admits a natural cocycle $\omega : \mathcal{R} \to \Gamma$ satisfying $\omega(gx, x) = g$ for all $g \in \Gamma$ and a.e. $x \in G$.

We will need the following easy result on the image of cocycles.

Lemma 2.5.12. *Let G be a locally compact group and $\mathcal{R}$ a countable equivalence relation on (X, μ) with μ a finite measure. Let $\omega : \mathcal{R} \to G$ be a cocycle. Then, for every locally bounded set $\mathcal{W} \subseteq \mathcal{R}$ and every $\varepsilon > 0$, there exists a Borel subset $E \subseteq X$ with $\nu(X \setminus E) < \varepsilon$ such that $\omega\big(\mathcal{W} \cap (E \times E)\big)$ is relatively compact.*

Proof. Since every bounded Borel subset can be written as a finite union of graphs of elements in $[[\mathcal{R}]]$, it suffices to prove (b) for graph(φ) with $\varphi \in [[\mathcal{R}]]$. But, if $(K_n)_n$ is an increasing sequence of compact sets with $G = \bigcup_n K_n$, then for

$$E_n = \left\{x \in X \;\middle|\; \omega\big(\varphi(x), x\big) \in K_n\right\},$$

we have $X = \bigcup E_n$. Since $\lim_n \mu(X \setminus E_n) = 0$, we find the required set by taking n large enough. □

2.5.2 Types of countable equivalence relations

Just as with factors of von Neumann algebras, countable equivalence relations can be classified into three types based on the existence of *invariant* measures. The following result follows directly from (2.5.1) (see also [FM77a, Corollary 1]).

Proposition 2.5.13. *Let $\mathcal{R}$ be a countable equivalence relation on (X, μ). Then, the following are equivalent.*

(i) *The left and right counting measures on $\mathcal{R}$ coincide.*

(ii) *For any partial Borel automorphism $\varphi : A \to B$ in $[[\mathcal{R}]]$, we have $\mathrm{d}\varphi_*\mu|_A = \mathrm{d}\mu|_B$.*

(iii) *There are partial Borel automorphisms $\varphi_i : A_i \to B_i$ such that $\mathcal{R} = \bigcup_i \operatorname{graph} \varphi_i$ and $\mathrm{d}\varphi_*\mu_i|_{A_i} = \mathrm{d}\mu|_{B_i}$ for all i.*

If μ satisfies these equivalent conditions, then μ is said to be invariant *for $\mathcal{R}$.*

A countable equivalence relation $\mathcal{R}$ admitting an invariant measure is called *measure preserving (mp)*. If $\mathcal{R}$ admits an invariant *probability* measure, then $\mathcal{R}$ is called *probability measure preserving (pmp)*. Just as with traces on factors, for an ergodic equivalence relation $\mathcal{R}$ on (X, μ), there can be at most one measure μ_0 (up to scaling) in the same measure class as μ that is invariant for $\mathcal{R}$.

The type classification is now as follows (see also [FM77a, Definition 3.4]).

Definition 2.5.14. Let $\mathcal{R}$ be an ergodic countable equivalence relation.

(a) $\mathcal{R}$ is of *type I* it is isomorphic to the full equivalence relation $\mathcal{R} = \{(x, y)\}_{x,y \in X}$ on a countable set X. If $|X| = n$ for $n \in \{1, \ldots, \infty\}$, we say that $\mathcal{R}$ is of *type I_n*.

(b) $\mathcal{R}$ is of *type II* if it is not of type I and there exists an invariant measure μ_0 in the measure class of μ. If μ_0 is finite, then we say that $\mathcal{R}$ is of *type II_1*, otherwise we say that $\mathcal{R}$ is of type II_∞.

(c) $\mathcal{R}$ is of *type III* in all other cases.

2.5.3 The von Neumann algebra associated to a countable equivalence relation

In this section, we discuss the construction of a von Neumann algebra associated to an equivalence relation. We will see that is construction is a generalization of the group measure space construction for free actions of countable groups. This construction was first introduced in [FM77b].

Let $\mathcal{R}$ be a countable equivalence relation on (X, μ). Endow $\mathcal{R}$ with the right counting measure. Following [FM77b], we call a Borel function $F : \mathcal{R} \to \mathbb{C}$ *(left) finite* if it is bounded and if its support is a bounded Borel set. We denote the set of left finite functions by $\mathcal{M}_f(\mathcal{R})$. For every $F \in \mathcal{M}_f(\mathcal{R})$, we define the operator

$$L_F : L^2(\mathcal{R}) \to L^2(\mathcal{R}) : (L_F\xi)(x, y) = \sum_{\substack{z \in X \\ z \sim x}} F(x, z)\xi(z, y).$$

This operator is bounded (see [FM77b, Proposition 2.1]) and if we equip $\mathcal{M}_f(\mathcal{R})$ with the following operations

$$(\lambda F_1 + \mu F_2)(x, y) = \lambda F_1(x, y) + \mu F_2(x, y)$$

$$(F_1 * F_2)(x, y) = \sum_{z \sim x} F_1(x, z)F_2(z, y)$$

$$F^*(x, y) = \overline{F(y, x)}$$

for $F, F_1, F_2 \in \mathcal{M}_f(\mathcal{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then the map $f \mapsto L_f$ becomes an injective $*$-morphism. The von Neumann algebra associated to a countable equivalence relation is now defined as follows.

Definition 2.5.15. The *von Neumann algebra* $L(\mathcal{R})$ *associated to a countable equivalence relation* $\mathcal{R}$ is the von Neumann algebra generated by the operators $\{L_F\}_{F \in \mathcal{M}_f(\mathcal{R})}$, i.e.

$$L(\mathcal{R}) = \overline{\{L_F\}_{F \in \mathcal{M}_f(\mathcal{R})}}^{\text{w.o.}}.$$

The von Neumann algebra $L(\mathcal{R})$ contains a canonical copy of $L^\infty(X)$ by identifying an $f \in L^\infty(X)$ with the operator associated to the function $F(x, y) = f(x)1_\Delta(x, y)$ in $M_f(\mathcal{R})$, where 1_Δ denotes the characteristic function of the diagonal $\Delta = \{(x, x)\}_{x \in X}$. The von Neumann algebra $L(\mathcal{R})$ also contains a copy of the full pseudogroup by identifying a partial Borel automorphism $\varphi : A \to B$ in $[[\mathcal{R}]]$ with the operator u_φ associated to the function $F(x, y) = 1_{\text{graph}\,\varphi}(x, y)$, where as before $\text{graph}\,\varphi = \{\big(\varphi(x), x\big)\}_{x \in X}$. Note that the action of these operators on $L^2(\mathcal{R})$ is given by

$$(L_f\xi)(x, y) = f(x)\xi(x, y) \qquad \text{and} \qquad (u_\varphi\xi)(x, y) = \xi\big(\varphi^{-1}(x), y\big)1_B(x)$$

for all $f \in L^\infty(X)$, all $\varphi : A \to B$ in $[[\mathcal{R}]]$, all $\xi \in L^2(\mathcal{R})$ and a.e. $(x, y) \in \mathcal{R}$. Moreover, $u_\varphi L_f u_\varphi^* = L_{f \circ \varphi^{-1}}$ for $f \in L^\infty(X)$ and $\varphi \in [\mathcal{R}]$.

By [FM77b, Propositions 2.3 and 2.4], every $F \in \mathcal{M}_f(\mathcal{R})$ can be written as a finite sum $F = \sum f_\varphi 1_{\mathrm{graph}\,\varphi}$ where $\varphi \in [\mathcal{R}]$ and $f_\varphi \in L^\infty(X)$. It follows that $L(\mathcal{R})$ is generated by $L^\infty(X)$ and $\{u_\varphi\}_{\varphi\in\mathcal{R}}$. Since $u_\varphi = L_{1_E}$ for $\varphi = \mathrm{id}_E$ and $E \subseteq X$ measurable, and the characteristic functions generate $L^\infty(X)$, we even have that $L(\mathcal{R})$ is generated by the operators $\{u_\varphi\}_{\varphi\in[[\mathcal{R}]]}$.

As we will see later, the subalgebra $L^\infty(X)$ plays a special role in $L(\mathcal{R})$. The following result can be found in [FM77b, Proposition 2.9].

Theorem 2.5.16. *Let $\mathcal{R}$ be a countable equivalence relation on (X, μ). Denote by $A = L^\infty(X)$. Then,*

- *(a) A is a maximal abelian subalgebra of $L(\mathcal{R})$, i.e. $A' \cap L(\mathcal{R}) = A$,*
- *(b) $L(\mathcal{R})$ is generated by the normalizer $\mathcal{N}_M(A)$,*
- *(c) there exists a normal faithful conditional expectation $E : L(\mathcal{R}) \to A$.*

As we will see in the next chapter, this means that A is a *Cartan subalgebra*.

Moreover, the following characterizes when $L(\mathcal{R})$ is a factor. A proof can again be found in [FM77b, Proposition 2.9 (2)].

Theorem 2.5.17. *Let $\mathcal{R}$ be a countable equivalence relation on (X, μ). Denote by $A = L^\infty(X)$. Then, the center of $L(\mathcal{R})$ is the algebra $A^{\mathcal{R}}$ of invariant functions on $\mathcal{R}$. In particular, $L(\mathcal{R})$ is a factor if and only if $\mathcal{R}$ is ergodic.*

The construction of $L(\mathcal{R})$ is a generalization of the group measure space construction for countable groups in the following way.

Proposition 2.5.18. *Let $\Gamma \curvearrowright (X, \mu)$ be a free action of a countable discrete group Γ. Denote $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$. Then,*

$$L(\mathcal{R}) \cong L^\infty(X) \rtimes \Gamma.$$

If μ is a finite measure, then $L(\mathcal{R})$ carries a natural state φ induced by the measure μ (see [Tak03b, Theorem XIII.2.11]). It satisfies

$$\varphi(L_F) = \int_X F(x, x)\,\mathrm{d}\mu(x)$$

for all $F = G * H$ with $G, H \in \mathcal{M}_f(\mathcal{R})$ such that $D^n G, D^n H \in L^2(\mathcal{R}, \nu_r)$ for all $n \in \mathbb{N}$. Here, D denotes the Radon-Nikodym derivative. The associated modular conjugation J is given by

$$(J\xi)(x, y) = \overline{\xi(y, x)} D(x, y)^{1/2}$$

for $\xi \in L^2(\mathcal{R})$ and a.e. $(x,y) \in \mathcal{R}$, and the modular automorphism group is given by

$$\sigma_t(L_F) = L_{FD^{it}}$$

for every $F \in \mathcal{M}_f(\mathcal{R})$. In particular, φ is a trace if μ is invariant.

The type of the countable equivalence relation is compatible with the type of the associated von Neumann algebra. A proof can be found in [Tak03b, Theorem XIII.2.10].

Theorem 2.5.19. *Let $\mathcal{R}$ be an ergodic countable equivalence relation. Then, $\mathcal{R}$ is of type I, II_1, II_∞ or III if and only if $L(\mathcal{R})$ is.*

The Maharam extension

In this section we discuss the Maharam extension, which associates to every (not necessarily measure preserving) countable equivalence relation $\mathcal{R}$ a measure preserving countable equivalence relation $\widetilde{\mathcal{R}}$. The construction is due to Maharam in [Mah64]. As we will see, this construction is closely related to the continuous core from Section 2.4.5.

Let $\mathcal{R}$ be a countable equivalence relation on (X,μ). By removing a conull set from X, we can assume that the Radon-Nikodym cocycle D satisfies the cocycle equation $D(x,y)D(y,z) = D(x,z)$ for all $(x,y,z) \in \mathcal{R}^{(2)}$ (see [Zim84, Theorem B.9]). The Maharam extension is defined as follows.

Definition 2.5.20. Let $\mathcal{R}$ be any countable equivalence relation on a standard measure space (X,μ). The *Maharam extension* of $\mathcal{R}$ is the equivalence relation $\widetilde{\mathcal{R}}$ on $X \times \mathbb{R}$ given by

$$(x,\lambda) \sim_{\widetilde{\mathcal{R}}} (y,\mu) \quad \text{if and only if} \quad x \sim_{\mathcal{R}} y \text{ and } \lambda = \log(D(x,y)) + \mu,$$

where D is a Radon-Nikodym cocycle.

The definition of the Maharam extension is precisely such that it carries a natural invariant measure.

Theorem 2.5.21. *Let $\mathcal{R}$ be any countable equivalence relation on a standard measure space (X,μ) and let $\widetilde{\mathcal{R}}$ be its Maharam extension. Then, the measure $\mathrm{d}\mu \otimes \exp(-t)\,\mathrm{d}t$ is invariant under $\widetilde{\mathcal{R}}$.*

The relation with the continuous core of von Neumann algebras is now as follows.

Theorem 2.5.22. *Let $\mathcal{R}$ be any countable equivalence relation on a standard measure space (X, μ). Denote by φ the canonical weight on $L(\mathcal{R})$ induced by the measure μ. Then there exists a $*$-isomorphism*

$$\Phi : c_\varphi\big(L(\mathcal{R})\big) \to L(\widetilde{\mathcal{R}})$$

such that $\Phi\big(A \rtimes_\varphi \mathbb{R}\big) = L^\infty(X \times \mathbb{R})$.

2.5.4 Cartan subalgebras

The algebra $A = L^\infty(X)$ in the von Neumann algebra $M = L(\mathcal{R})$ associated to a countable equivalence relation $\mathcal{R}$ on (X, μ) plays a special role. Indeed, due to the following theorem of Singer from [Sin55], the inclusion $A \subseteq M$ contains all information of the relation $\mathcal{R}$.

Theorem 2.5.23 (Singer's theorem)**.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be a countable equivalence relation on (X_1, μ_1) and (X_2, μ_2) respectively. Then, there exists a $*$-isomorphism $\Phi : L(\mathcal{R}_1) \to L(\mathcal{R}_2)$ satisfying $\Phi\big(L^\infty(X_1)\big) = L^\infty(X_2)$ if and only if $\mathcal{R}_1 \cong \mathcal{R}_2$.*

Moreover, using the form of Singer's theorem in [FM77b, Theorem 2], the $*$-isomorphism π is of the following form: there exists a isomorphism $\theta : X_1 \to X_2$ of countable equivalence relations from $\mathcal{R}_1$ to $\mathcal{R}_2$ and a cocycle $\omega : \mathcal{R}_1 \to \mathbb{T}$ such that for all $f \in L^\infty(X)$ and all $\varphi \in [\mathcal{R}]$, we have

$$\pi(f u_\varphi) = \theta_*(\omega_\varphi f) u_\psi,$$

where $\psi = \theta \circ \varphi \circ \theta^{-1}$, the $*$-morphism $\theta_* : L^\infty(X_1) \to L^\infty(X_2)$ is given by $\theta_*(f) = f \circ \theta^{-1}$ and $\omega_\varphi \in L^\infty(X_1)$ is given by $\omega_\varphi(x) = \omega\big(x, \varphi^{-1}(x)\big)$.

Inclusions of the form $L^\infty(X) \subseteq L(\mathcal{R})$ can be axiomatized in the following way.

Definition 2.5.24. Let M be a von Neumann algebra. A subalgebra $A \subseteq M$ is said to be a *Cartan subalgebra* if

(i) A is a *maximial abelian subalgebra (MASA)*, i.e. $A' \cap M = A$,

(ii) A is *regular*, i.e. its *normalizer* $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M,

(iii) A is with expectation, i.e. there exists a normal, faithful conditional expectation $E : M \to A$.

By Theorem 2.5.16, the subalgebra $L^\infty(X)$ is a Cartan subalgebra of $L(\mathcal{R})$. Moreover, Feldman and Moore proved in [FM77b, Theorem 1] that, every inclusion $A \subseteq M$ of a Cartan subalgebra A in a von Neumann algebra M 'almost' is of this form. For the slightly more general construction of the von Neumann algebra $L(\mathcal{R}, \omega)$ for a cocycle $\omega : \mathcal{R} \to \mathbb{T}$, see [FM77b].

Theorem 2.5.25. *Let M be a von Neumann algebra and $A \subseteq M$ a Cartan subalgebra. Then, there exists a countable equivalence relation on a standard Borel space (X, μ), a cocycle $\omega : \mathcal{R} \to \mathbb{T}$ and a $*$-isomorphism $\Phi : M \to L(\mathcal{R}, \omega) \to M$ such that $\Phi(A) = L^\infty(X)$. Moreover, $\mathcal{R}$ is unique up to isomorphism.*

As explained in the introduction, uniqueness of Cartan subalgebras plays an important role in studying rigidity results of crossed product.

Definition 2.5.26. Let M be a von Neumann algebra with a Cartan subalgebra $A \subseteq M$. We say that

(a) A is *unique up to conjugacy* if for every other Cartan subalgebra $B \subseteq M$, there exists an automorphism $\Phi \in \text{Aut}(M)$ with $\Phi(A) = B$,

(b) A is *unique up to unitary conjugacy* if for every other Cartan subalgebra $B \subseteq M$, there exists a unitary $u \in \mathcal{U}(M)$ such that $B = uAu^*$.

Singer's theorem now yields the following.

Corollary 2.5.27. *Let $\mathcal{R}$ be a countable equivalence relation. Suppose that $L(\mathcal{R})$ has unique Cartan subalgebra up to conjugacy. Then, for every other countable equivalence relation $\mathcal{S}$, we have $L(\mathcal{R}) \cong L(\mathcal{S})$ if and only if $\mathcal{R} \cong \mathcal{S}$.*

We end this section by the following results on the relation between Cartan subalgebras of M and those of its corners pMp. All of the results below are well-know, but for completeness we include proofs.

Lemma 2.5.28. *Let M be a von Neumann algebra with a Cartan subalgebra $A \subseteq M$. Let $p \in A$ be a projection. Then, Ap is a Cartan subalgebra of pMp.*

Proof. By Theorem 2.5.25, we can assume that $M = L(\mathcal{R})$ and $A = L^\infty(X)$ for some countable equivalence relation $\mathcal{R}$ on (X, μ). Now, the projection $p \in A$ is of the form 1_E for some measurable $E \subseteq X$. It follows that $pMp = L(\mathcal{R}|_E)$ and $Ap = L^\infty(E)$. ☐

Also the following lemma is well-known to experts. Proofs for the finite case and the semifinite case can for instance be found in [BO08, Corollary F.8] and

[HV13, Lemma 2.1]. For the readers convenience, we show how to deduce the general case from these two results.

Lemma 2.5.29. *Let M be a von Neumann algebra and $A \subseteq M$ a maximal abelian subalgebra with expectation. Then, for every projection $p \in M$, there exists a projection $q \in A$ with $q \sim p$.*

Proof. Using for instance [Tak03a, Theorem V.1.19], we can write $p = p_1 + p_2$ for projections $p_1, p_2 \in M$ with orthogonal central support and such that p_1 is finite and p_2 is properly infinite. Denote by z_1 and z_2 the central supports of p_1 and p_2 respectively. By [Tak03a, Theorem V.1.39], we have $p_2 \sim z_2$. Moreover, Mz_1 is semifinite. Let $\mathrm{Tr} : (Mz_1)^+ \to [0, +\infty]$ be a faithful, normal semifinite trace. By [Tak03a, Lemma V.7.11], $\mathrm{Tr}\,|_{Az_1}$ is also semifinite and hence, by [HV13, Lemma 2.1], there exists a projection $q_1 \in Az_1$ such that $p_1 \sim q_1$. Taking $q = q_1 + z_2$ yields the result. □

The relation between Cartan subalgebras of M and its corners is now given as follows. Again, for the readers convenience, we include a proof of this result.

Proposition 2.5.30. *Let M be a von Neumann algebra and $p \in M$ a projection with central support 1. Then, there is a 1-1 correspondence between the equivalence classes of Cartan subalgebras of M up to unitary conjugacy, and the equivalence classes of Cartan subalgebras of pMp up to unitary conjugacy.*

Proof. Suppose that $M \subseteq B(\mathcal{H})$. Using Zorn's lemma, we take a maximal orthogonal family of projections $\{q_i\}_{i\in I}$ such that $q_i \preceq p$. Since p has central support 1, we have $\sum_i q_i = 1$ (see for instance [AP14, Lemma 2.4.6]). Take partial isometries $u_i \in M$ such that $p_i = u_i^* u_i \leqslant p$ and $q_i = u_i u_i^*$. Denote by $\mathcal{K} = \ell^2(I)$ and let $(e_i)_i$ be the canonical orthonormal basis. Let $U : \mathcal{H} \to p\mathcal{H} \otimes \mathcal{K}$ be the isometry defined by

$$U\xi = \sum_i u_i \xi \otimes e_i$$

for $\xi \in \mathcal{H}$. Take $\tilde{p} = UU^*$. Then,

$$\Phi : M \to \tilde{p}\big(pMp \mathbin{\overline{\otimes}} B(\mathcal{K})\big)\tilde{p} : x \mapsto UxU^*$$

is a $*$-isomorphism. Denote $D = \ell^\infty(I)$.

The 1-1 correspondence is now defined as follows. Let $\mathcal{A}$ be an equivalence class of Cartan subalgebras in M. Take $A \in \mathcal{A}$. By Lemma 2.5.29, we can take a projection $r \in A$ with $r \sim p$. Let $v \in M$ be a partial isometry with $p = v^* v$ and $r = vv^*$. By Lemma 2.5.28, Ar is a Cartan subalgebra in rMr.

Using the $*$-isomorphism $rMr \to pMp : x \mapsto v^*xv$, we see that $B = v^*Av$ is a Cartan subalgebra of pMp. Note that the equivalence class of B up to unitary conjugacy does not depend on the choice of r and v, or on the choice of representant A. Indeed, let $A_1 \in \mathcal{A}$ be another representant. Take a unitary $u \in M$ such that $A_1 = uAu^*$ Let $r_1 \in A_1$ be a projection with $r_1 \sim p$ and let $v_1 \in M$ be such that $p = v_1^*v_1$ and $r_1 = v_1v_1^*$. Then, $w = v_1^*uv$ is a unitary in pMp such that $w(v^*Av)w^* = v_1^*A_1v_1$.

Conversely, let $\mathcal{B}$ be an equivalence class of Cartan subalgebras in pMp. Take $B \in \mathcal{B}$. Since $D \subseteq B(\mathcal{H})$ is a Cartan subalgebra, it follows that $B \otimes \ell^\infty(I)$ is a Cartan subalgebra of $pMp \mathbin{\overline{\otimes}} B(\mathcal{H})$. Take a projection $\tilde{r} \in B \mathbin{\overline{\otimes}} D$ with $\tilde{r} \sim \tilde{p}$ and a partial isometry $\tilde{v} \in pMp \mathbin{\overline{\otimes}} B(\mathcal{H})$ such that $\tilde{p} = \tilde{v}^*\tilde{v}$ and $\tilde{r} = \tilde{v}\tilde{v}^*$. As before, we have that $A = \Phi^{-1}\big(\tilde{v}^*(B \mathbin{\overline{\otimes}} D)\tilde{v}\big)$ is a Cartan subalgebra of M. Let $B_1 \in \mathcal{B}$ is another representant, say with $B_1 = uBu^*$ for some unitary $u \in pMp$. Take a projection $\tilde{r}_1 \in B_1 \mathbin{\overline{\otimes}} D$ with $\tilde{r}_1 \sim \tilde{p}$ and let $\tilde{v}_1 \in pMp \mathbin{\overline{\otimes}} B(\mathcal{H})$ be such that $\tilde{p} = \tilde{v}_1^*\tilde{v}_1$ and $\tilde{r}_1 = \tilde{v}_1\tilde{v}_1^*$. Then, $\tilde{w} = \Phi^{-1}(\tilde{v}_1^*(u \otimes 1)\tilde{v})$ is a unitary in $pMp \mathbin{\overline{\otimes}} B(\mathcal{H})$ implements the unitary conjugacy between A and $A_1 = \Phi^{-1}\big(\tilde{v}_1^*(B_1 \mathbin{\overline{\otimes}} D)\tilde{v}_1\big)$.

One checks that if $A \subseteq M$ is a Cartan subalgebra, then A is unitarily conjugated with $A_1 = \Phi^{-1}\big(\tilde{v}^*(v^*Av \mathbin{\overline{\otimes}} D)\tilde{v}\big)$ and conversely that if B is a Cartan subalgebra in Mp, then B is unitarily conjugate to $v^*\Phi^{-1}\big(\tilde{v}^*(B \mathbin{\overline{\otimes}} D)\tilde{v}\big)v$, where and $v \in M$ and $\tilde{v} \in pMp \mathbin{\overline{\otimes}} B(\mathcal{H})$ is are partial isometries chosen as above.

□

We also need the result on the relation between a Cartan subalgebra of M and the Cartan subalgebra of the continuous core $c_\varphi(M)$.

Proposition 2.5.31. *Let M be a von Neumann algebra and $A \subseteq M$ a Cartan subalgebra. Denote by $E : M \to A$ a conditional expectation. Let φ be a state on A and write $\tilde{\varphi} = \varphi \circ E$. Then, $c_\varphi(A) = A \rtimes_{\sigma^\varphi} \mathbb{R}$ is a Cartan subalgebra of $c_\varphi(M)$.*

Proof. By Theorem 2.5.25, we can assume that $M = L(\mathcal{R})$ and $A = L^\infty(X)$ for some countable equivalence relation $\mathcal{R}$ on (X, μ). Moreover, we can assume that μ is the probability measure that induced the state φ. Now, the result follows immediately by Theorem 2.5.22, since $L^\infty(X \times \mathbb{R})$ is a Cartan subalgebra in $L(\widetilde{\mathcal{R}})$. □

Unitary conjugacy and Popa's intertwining-by-bimodules

Popa's concept of intertwining-by-bimodules, introduced in [Pop06b], is a key tool in deformation/rigidity theory. We will use this method to prove unitary conjugacy of Cartan subalgebras. The method can be used more generally to "locate" certain subalgebras in tracial von Neumann algebras. The central theorem is the following. It was obtained by Popa in [Pop06b, Theorem 2.1 and Corollary 2.3].

Theorem 2.5.32. *Let (M, τ) be a tracial von Neumann algebra. Let $p, q \in M$ be projections and $P \subseteq pMp$ and $Q \subseteq qMq$ be von Neumann subalgebras. Then, the following are equivalent.*

(i) *There exists a nonzero projection $r \in M_n(\mathbb{C}) \otimes Q$, a normal unital $*$-morphism $\theta : P \to r(M_n(\mathbb{C}) \otimes Q)r$ and a nonzero partial isometry $v \in M_{1,n}(\mathbb{C}) \otimes pMq$ such that $xv = v\theta(x)$ for all $x \in P$.*

(ii) *There exist nonzero projections $p_0 \in P$ and $q_0 \in Q$, a normal unital $*$-morphism $\theta : p_0Pp_0 \to q_0Qq_0$ and a nonzero partial isometry $v \in p_0Mq_0$ such that $xv = v\theta(x)$ for all $x \in p_0Pp_0$.*

(iii) *There exists no net of unitaries $(u_i)_i$ in P satisfying $\|E_Q(x^*u_iy)\|_2 \to 0$ for all $x, y \in pMq$. Here, $E_Q : M \to Q$ denotes the unique trace-preserving conditional expectation.*

If one of these equivalent conditions hold, then we say that (a corner of) P intertwines into Q inside M *and we write $P \prec_M Q$.*

In [Pop06a], Popa generalized these intertwining techniques to the case when M is endowed with an almost periodic state φ and $A \subseteq pM^\varphi p$ and $B \subseteq qM^\varphi q$. Later, it was shown in [HV13] by Houdayer and Vaes that these techniques extend to the case when Q is finite and with expectation, and P and M are arbitrary. Ueda subsequently extended this to the case that Q is semifinite and with expectation in [Ued13]. In [HI17], Houdayer and Isono were able generalize the techniques the case where P is finite and with expectation and Q is arbitrary with expectation. Very recently, Isono succeeded in generalizing the techniques to arbitrary subalgebras P and Q with expectation in [Iso19].

Below, we will discuss the generalizations in [HV13] and [HI17]. Since we will not use the other generalizations, we will not discuss them here.

The following is a combination of [HV13, Theorem 2.3] and [HI17, Theorem 4.3].

Theorem 2.5.33. *Let M be a von Neumann algebra. Let $p, q \in M$ be projections and let $P \subseteq pMp$ and $Q \subseteq qMq$ be von Neumann subalgebras with expectation. Assume that P or Q is finite. Then, the following are equivalent.*

(i) *There exists a nonzero projection* $r \in M_n(\mathbb{C}) \otimes Q$*, a normal unital* $*$*-morphism* $\theta : P \to r\big(M_n(\mathbb{C}) \otimes Q\big)r$ *and a nonzero partial isometry* $v \in M_{1,n}(\mathbb{C}) \otimes pMq$ *such that* $xv = v\theta(x)$ *for all* $x \in P$ *and such that the inclusion* $\theta(P) \subseteq r\big(M_n(\mathbb{C}) \otimes Q\big)r$ *is with expectation.*

(ii) *There exist a nonzero projections* $p_0 \in P$ *and* $q_0 \in Q$*, a normal unital* $*$*-morphism* $\theta : p_0Pp_0 \to q_0Qq_0$ *and a nonzero partial isometry* $v \in p_0Mq_0$ *such that* $xv = v\theta(x)$ *for all* $x \in p_0Pp_0$ *and such that* $\theta(p_0Pp_0) \subseteq q_0Qq_0$ *is with expectation.*

(iii) *There is no net of unitaries* $(u_i)_i$ *in* P *such that* $E_B(x^*u_iy) \to 0$ *in the* $*$*-strong operator topology for all* $x, y \in pMq$*.*

If one of these equivalent conditions hold, then we say that (a corner of) P intertwines into Q inside M *and we write* $P \prec_M Q$*.*

For a Cartan subalgebra intertwining into another Cartan subalgebra is equivalent to being unitarily conjugate. This key result was obtained by [Pop06a] in [Pop06a, Lemma A.1] for finite von Neumann algebras. The result presented here is the slight generalization of [HV13, Theorem 2.5] to arbitrary von Neumann algebras.

Theorem 2.5.34. *Let* M *be any von Neumann algebra and* $A, B \subseteq M$ *maximal abelian subalgebras with expectation. Consider the following assertions.*

(i) *There exists a unitary* $u \in M$ *such that* $uAu^* = B$*.*

(ii) $A \prec_M B$*.*

(iii) *There exists a nonzero partial isometry* $v \in M$ *such that* $v^*v \in A$*,* $vv^* \in B$ *and* $vAv^* = Bvv^*$*.*

Then, (i)⇒(ii)⇔(iii) and if A *and* B *are Cartan subalgebras and* M *is a factor, then (i)⇔(ii).*

2.5.5 Cross section equivalence relations

For a nonsingular action $G \curvearrowright (X, \mu)$ of a *noncountable* group on a standard measure space, the orbit equivalence relation $\mathcal{R}(G \curvearrowright X) = \{(gx, x) \mid g \in G, x \in X\}$ does *not* have countable orbits. However, we will see that by restricting $\mathcal{R}(G \curvearrowright X)$ to a so-called *cross section*, one does obtain a countable equivalence relation that still contains a lot of information about the original original action.

The notion of a cross section was introduced by Forrest in [For74]. A more recent and rather self-contained treatment for unimodular groups can be found in [KPV15, Section 4.1].

Definition 2.5.35. Let $G \curvearrowright (X, \mu)$ be a nonsingular action of a locally compact group G. A *cross section* is a Borel subset $X_1 \subseteq X$ with the following two properties.

(i) There exists a neighborhood $\mathcal{U} \subseteq G$ of identity such that the action map $\mathcal{U} \times X_1 \to X : (g, x) \mapsto gx$ is injective.

(ii) The subset $G \cdot X_1 \subseteq X$ is conull.

A *partial cross section* is a Borel subset $X_1 \subseteq X$ such that condition (i) is satisfied and such that $G \cdot X_1$ is nonnull.

Note that the first condition implies that the action map $\theta : G \times X_1 \to X : (g, x) \mapsto gx$ is countable-to-one and hence maps Borel sets to Borel sets (see [Kec95, p. 18.14]). In particular, the set $G \cdot X_1$ in the second condition is Borel.

Forrest obtained the following key result in [For74, Theorem 2.10]. A proof can also be found in [KPV15, Theorem 4.2].

Theorem 2.5.36. *Let $G \curvearrowright (X, \mu)$ be an essentially free, nonsingular action of a locally compact group G. Then, there exists a cross section $X_1 \subseteq X$.*

Let $G \curvearrowright (X, \mu)$ be an essentially free, nonsingular action. By removing a G-invariant null set from X, we can always assume that $G \cdot X_1 = X$ and that $G \curvearrowright X$ is really free. Hence, by [Kec95, 18.10 and 18.14], one can take a Borel map that is a right inverse of the action map θ. This yields Borel maps $\pi : X \to X_1$ and $\gamma : X \to G$ such that $x = \gamma(x) \cdot \pi(x)$ for all $x \in X$. If $\mathcal{U}$ is a neighborhood of identity such that the action map $\theta : \mathcal{U} \times X_1 \to X$ is injective, then we can take γ and μ such that $\gamma(gy) = g$ and $\pi(gx) = x$ for $g \in \mathcal{U}$ and $x \in X_1$.

Similarly, the map $G \times X \to X \times X : (g, x) \mapsto (gx, x)$ is injective. Hence, its image $\mathcal{R}(G \curvearrowright X)$ is Borel and it admits a Borel inverse. This yields a Borel map $\omega : \mathcal{R}(G \curvearrowright X) \to G$ satisfying $\omega(x, y)y = x$ for $y \in G \cdot x$. Moreover, ω satisfies the cocycle identity $\omega(x, y)\omega(y, z) = \omega(x, z)$ for all $y, z \in G \cdot x$. In particular, the restriction of ω to $\mathcal{R}$ is a cocycle in the sense of Definition 2.5.11.

The restriction of $\mathcal{R}(G \curvearrowright X)$ that we talked about in the beginning of this section is now the following.

Definition 2.5.37. Let $G \curvearrowright (X,\mu)$ be an essentially free, nonsingular action. Given a partial cross section $X_1 \subseteq X$, we call

$$\mathcal{R}(G \curvearrowright X) \cap (X_1 \times X_1) = \{(x,y) \in X_1 \times X_1 \mid \exists g \in G : x = gy\}$$

the *cross section equivalence relation* on X_1.

Following [Zim84, Example 4.2.4], one defines the *Radon-Nikodym cocycle* $D : G \cdot X \to \mathbb{R}_0^+$ for the action $G \curvearrowright X$ given by

$$D(g,x) = \frac{\mathrm{d}g^{-1} \cdot \mu}{\mathrm{d}\mu}(x)$$

for all $g \in G$ and a.e. $x \in X$. Moreover, we can assume that D is a Borel cocycle in the sense that $D(gh,x) = D(g,hx)D(h,x)$ for all $g,h \in G$ and a.e. $x \in X$. See [Zim84, Section 4.2] for more details on cocycles for measurable actions.

Defining $\omega_r : \mathcal{R} \to \mathbb{C}$ by $\omega_r(x,y) = D(\omega(x,y),y)$, where $\omega : \mathcal{R} \to G$ is as before, we get a cocycle $\omega : \mathcal{R} \to \mathbb{R}_0^+$.

The following result generalizes [KPV15, Theorem 4.3] to nonunimodular locally compact groups and non-pmp actions. For completeness, we included a proof of this result in Appendix A.

Proposition 2.5.38. *Let $G \curvearrowright (X,\mu)$ be a essentially free, nonsingular action. Let $X_1 \subseteq X$ be a partial cross section and denote by $\mathcal{R}$ its cross section equivalence relation.*

(a) There exists a unique measure μ_1 (up to scaling) on X_1 satisfying

$$(\lambda_G \otimes \mu_1)(\mathcal{W}) = \int_X \sum_{\substack{(g,y)\in\mathcal{W} \\ x=gy}} D(g^{-1},x)\,\mathrm{d}\mu(x)$$

for all measurable $\mathcal{W} \subseteq G \times X_1$. Moreover, $\mathcal{R}$ is nonsingular for μ_1.

(b) $\mathcal{R}$ is nonsingular for μ_1 and its Radon-Nikodym cocycle is given by

$$D_1(x,y) = \delta_G\big(\omega(x_1,x_2)\big)\omega_r(x_1,x_2)$$

In particular, if G is unimodular and $G \curvearrowright (X,\mu)$ is measure preserving, then μ_1 is an invariant measure for $\mathcal{R}$.

(c) If X_1 is a cross section, then $\mathcal{R}$ is ergodic if and only if $G \curvearrowright (X,\mu)$ is ergodic.

Remark 2.5.39. If $G \curvearrowright (X,\mu)$ is pmp, then μ_1 is a finite measure. Indeed, if $\mathcal{U} \subseteq G$ is a neighborhood of unity such that $\theta : \mathcal{U} \times X_1 \to X$ is injective, then $\mu_1(X_1) = \mu(\mathcal{U} \cdot X_1)/\lambda_G(\mathcal{U}) < +\infty$. In this case, it is customary to normalize μ_1 such that it becomes a probability measure. Then, we have

$$(\lambda_G \otimes \mu_1)(\mathcal{W}) = \operatorname{covol}(X_1) \int_X |\mathcal{W} \cap \theta^{-1}(x)| \, \mathrm{d}\mu(x)$$

for all measurable $\mathcal{W} \subseteq G \times X_1$, where the scaling constant $\operatorname{covol}(X_1)$ is called the *covolume* of X_1.

We will need the following easy lemma in the rest of this thesis.

Lemma 2.5.40. *Let G be a locally compact group and $G \curvearrowright (X,\mu)$ an essentially free action. Let $X_1 \subseteq X$ be a partial cross section and $\mathcal{R}$ the associated cross section equivalence relation. Then,*

(a) *If $K \subseteq G$ is compact, then the set $\mathcal{W} = \{(x,y) \in \mathcal{R} \mid \omega(x,y) \in K\}$ is a bounded subset of $\mathcal{R}$.*

(b) *If $G \curvearrowright (X,\mu)$ is pmp, then for every locally bounded $\mathcal{W} \subseteq \mathcal{R}$ and every $\varepsilon > 0$, then there exists a Borel subset $E \subseteq X_1$ with $\nu(E) < \varepsilon$ such that $\omega\big(\mathcal{W} \cap (E \times E)\big)$ is relatively compact.*

Proof. Statement (a) follows easily from the fact that there is a neighborhood of the unit $e \in G$ for which the map $\mathcal{U} \times X_1 \to X : (g,x) \mapsto gx$ is injective. Statement (b) follows immediately from Lemma 2.5.12. □

Recall that for nonsingular, free actions $\Gamma \curvearrowright (X,\mu)$ of discrete groups, the von Neumann algebra associated to the orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ is isomorphic to the crossed product $L^\infty(X) \rtimes \Gamma$ (see Proposition 2.5.18). For crossed products of nonsingular, free actions $G \curvearrowright (X,\mu)$ of *nondiscrete* locally compact groups, a corner of the crossed product $M = L^\infty(X) \rtimes G$ is isomorphic to an amplification of the von Neumann algebra associated to a cross section equivalence relation of $G \curvearrowright (X,\mu)$. A proof was provided in [KPV15, Lemma 4.5] in the case that G is unimodular and $G \curvearrowright (X,\mu)$ is pmp. For completeness, we also provide a full proof in Appendix A.

Proposition 2.5.41. *Let $G \curvearrowright (X,\mu)$ be an essentially free, nonsingular action. Denote by $M = L^\infty(X) \rtimes G$ and let X_1 be a partial cross section. Then,*

$$pMp \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big),$$

where $\mathcal{U}$ is a neighborhood of identity such that the action map $\mathcal{U} \times X_1 \to X$ is injective and $p = 1_{\mathcal{U} \cdot X_1} \in L^\infty(X)$. In particular, if X_1 is a cross section, then p has central support 1 in M.

When G is nondiscrete, the subalgebra $L^\infty(X)$ is not a Cartan subalgebra of the crossed product $M = L^\infty(X) \rtimes G$, since there exists no faithful, normal conditional expectation $M \to L^\infty(X)$. However, combining the previous decomposition with Proposition 2.5.30 yields the following.

Corollary 2.5.42. *Let $G \curvearrowright (X, \mu)$ be an essentially free, nonsingular action action. Then, the von Neumann algebra $M = L^\infty(X) \rtimes G$ has a Cartan subalgebra. Moreover, M has unique Cartan subalgebra if and only if $L(\mathcal{R})$ has unique Cartan subalgebra, where $\mathcal{R}$ is the cross section equivalence relation associated to a choice of cross section $X_1 \subseteq X$.*

2.6 Measure equivalence

Measure equivalence is an equivalence relation on groups that was introduced by Gromov in [Gro93] for countable, discrete groups in which groups that are similar from the viewpoint of measurable group theory are equivalent. It can be viewed as a measure-theoretic analogue of the notion of quasi-isometry in geometric group theory.

A notion of measure equivalence for locally compact groups was introduced by Bader, Furman, and Sauer in [BFS13] (in the unimodular case) and by S. Deprez and Li in [DL14] (in the nonunimodular case). More recently, this notion has been studied in detail by Koivisto, Kyed, and Raum in [KKR17; KKR18].

The following definition is taken from [DL14, Definition 3.1].

Definition 2.6.1. Let G and H be two locally compact groups. A *measure G-H-correspondence* is a standard measure space (Ω, η) together with a nonsingular action $G \times H \curvearrowright (\Omega, \eta)$ for which there exists

(i) a standard probability space (X, μ) and a nonsingular isomorphism $\varphi : X \times H \to \Omega$ satisfying $\varphi(x, hk) = h\varphi(x, k)$ for all $h \in H$ and a.e. $x \in X$ and $k \in H$,

(ii) a standard probability space (Y, ν) and a nonsingular isomorphism $\psi : Y \times G \to \Omega$ satisfying $\psi(y, gk) = g\psi(u, k)$ for all $g \in G$ and a.e. $y \in Y$ and $k \in G$.

Note that any two locally compact groups admit a measure G-H-correspondence. Indeed, take $\Omega = G \times H$, let $\eta = \lambda_G \otimes \lambda_H$ and let $G \times H \curvearrowright \Omega$ diagonally. Fixing a probability measure μ and ν on $X = G$ and $Y = H$ respectively in the same measure class as the Haar measure, we get nonsingular isomorphisms $\varphi : X \times H \to \Omega$ and $\psi : Y \times G \to \Omega$ as in the definition by defining

$$\varphi(g,h) = (g,h) \qquad \text{and} \qquad \psi(h,g) = (g,h)$$

for $g \in H$ and $h \in H$. This measure G-H-correspondence is called the *coarse G-H-correspondence.*

Given a measure G-H-correspondence (Ω, η), there exist unique nonsingular actions $G \curvearrowright X$ and $H \curvearrowright Y$ and Borel cocycles $\omega_G : H \times Y \to G$ and $\omega_H : G \times X \to H$ satisfying

$$g\varphi(x,h) = \varphi\big(gx, h\omega_H(g,x)^{-1}\big) \qquad \text{and} \qquad h\psi(y,g) = \psi\big(hy, h\omega_G(g,y)^{-1}\big)$$

for a.e. $x \in X$, $y \in Y$, $g \in G$ and $h \in H$. We call the actions $G \curvearrowright X$ and $H \curvearrowright Y$ the actions associated to Ω. See [DL14, Observation 3.3] for more details. Measure equivalence is now defined as follows.

Definition 2.6.2. Let G and H be two locally compact groups. A measure correspondence Ω is said to be a *measure equivalence coupling* if there exist probability measures μ' on X and ν' on Y in the same measure class as μ and ν respectively for which the actions $G \curvearrowright (X, \mu')$ and $H \curvearrowright (Y, \nu')$ associated to Ω are pmp.

If G and H admit a measure equivalence coupling, then G and H are said to be *measure equivalent.*

Example 2.6.3. Let G be a locally compact group and H a closed subgroup with $\delta_G|_H = \delta_H$. By the quotient integral formula (see for instance [DE14, Theorem 1.5.3]), the space G/H admits a G-invariant measure which is unique up to scaling. If this measure is finite (which for instance happens when H is cocompact or when H is a lattice in G), then G and H are measure equivalent. Indeed, if we equip (G, λ_G) with the action $G \times H \curvearrowright G$ given by $(g,h)k = gkh^{-1}$, then (G, λ_G) is a measure equivalence coupling. Fixing a Borel section $\sigma : G/H \to G$ for the quotient map $p : G \to G/H$, we can take $X = G/H$, $Y = \{y_0\}$ and $\varphi : X \times H \to G$ and $\psi : Y \times G \to G$ with

$$\varphi(x,h) = \sigma(x)h^{-1} \qquad \text{and} \qquad \psi(y_0, g) = g$$

for $g \in G$, $h \in H$ and $x \in X$.

Given essentially free, nonsingular actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \eta)$ of locally compact groups G, H, we say that the actions are *stably orbit equivalent* if they admit cross sections such that the associated cross section equivalence relations are isomorphic. Koivisto, Kyed, and Raum proved the following in [KKR18, Theorem A] and [KKR17, Theorem A].

Theorem 2.6.4. *Let G and H be locally compact groups. Then, the following are equivalent.*

(i) G *and* H *are measure equivalent.*

(ii) G *and* H *admit stably orbit equivalent, free, ergodic, pmp actions* $G \curvearrowright (X, \mu)$ *and* $H \curvearrowright (Y, \nu)$

Chapter 3

Ozawa's class $\mathcal{S}$ for locally compact groups

Recall from the introduction that class $\mathcal{S}$ for countable discrete groups was introduced by Ozawa in [Oza06] as the class of all countable discrete groups that are exact and that admit a map $\eta : \Gamma \to \mathrm{Prob}(\Gamma)$ satisfying

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

for all $g, h \in \Gamma$. By [Oza06, Theorem 4.1], this class can also be characterized as the class of groups that admit a topologically amenable action on a boundary that is *small at infinity* (in the sense of Definition 3.2.1 below). As mentioned in the introduction, class $\mathcal{S}$ is used for proving rigidity results of group von Neumann algebras and group measure space von Neumann algebras.

In this chapter, we define class $\mathcal{S}$ for locally compact groups as the class of groups that are exact and that admit a $\|.\|$-continuous map $\eta : G \to \mathrm{Prob}(G)$ satisfying

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0 \tag{3.0.1}$$

uniformly on compact sets for $g, h \in G$ (see also Definition A from the introduction). In Section 3.1, we prove some basic properties of (the groups in) this class in Section 3.1. Denoting by $\beta^{lu}G$ the left equivariant Stone-Čech compactification of G (see Section 2.3.1), we prove that for an exact group G, the existence of a map $\eta : G \to \mathrm{Prob}(G)$ as in (3.0.1) is equivalent to the existence of a map $\eta : G \to \mathrm{Prob}(\beta^{lu}G)$ satisfying the same equation (Theorem 3.1.7).

Next, in Section 3.2, we prove Theorem B from the introduction, i.e. we characterize locally compact groups in class $\mathcal{S}$ as groups having a topologically amenable action on a compactification that is small at infinity. This generalizes [Oza06, Theorem 4.1] to the locally compact setting. The two main novelties in this result are the fact that we prove that the action on the whole compactification h^uG is topologically amenable (in stead of only the action on the boundary) and the fact that we use another method to prove that the existence of such an action implies that the group belongs to class $\mathcal{S}$. Indeed, the original proof of Ozawa in the countable setting, used that G belongs to class $\mathcal{S}$ if and only if G is exact and satisfies the following Akemann-Ostrand-like property [AO75]: there exists a u.c.p. map $\theta : \mathrm{C}^*_r(G) \otimes_{\min} \mathrm{C}^*_r(G) \to B\big(L^2(G)\big)$ such that $\theta(a \otimes b) - \lambda(a)\rho(b) \in K\big(L^2(G)\big)$. Here, λ and ρ denote the representations of $\mathrm{C}^*_r(G)$ induced by the left and right regular representation respectively. However, this characterization of class $\mathcal{S}$ does not hold for general locally compact groups. Indeed, such a map θ for instance exists for all connected groups G, since $\mathrm{C}^*_r(G)$ is nuclear.

In Section 3.3, we prove that class $\mathcal{S}$ is a measure equivalence invariant (Theorem E), generalizing a result of Sako [Sak09a] to the locally compact setting. As explained in the introduction, the fact that exactness is preserved under measure equivalence follows from [DL15, Corollary 2.9] and [DL14, Theorem 0.1 (6)]. To prove that property (S) (i.e. the existence of a map $\eta : G \to \mathrm{Prob}(G)$ as in (1.3.1)) is preserved under measure equivalence, we define a notion of property (S) for countable equivalence relations (Definition 3.3.2) and we prove that a group G has property (S) if and only if a (and hence every) cross section equivalence relation of a (and hence every) free, ergodic, pmp action $G \curvearrowright (X, \mu)$ has this notion of property (S) (Proposition 3.3.6). Then, we conclude by using the characterization of measure equivalence in terms of isomorphism of cross section equivalence relations (see Theorem 2.6.4). As a consequence of this result, we have for instance that $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ are in class $\mathcal{S}$, since $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a lattice in both groups and belongs to $\mathcal{S}$ by [Oza09].

Examples of groups in class $\mathcal{S}$ include are amenable groups (Corollary 3.1.6) and connected, simple Lie groups of real rank one with finite center ([Ska88, Proof of Téorème 4.4]). In Section 3.4, we provide other examples of groups in class $\mathcal{S}$ by proving Proposition C and Theorem D from the introduction: we prove that groups admitting a proper action on a tree or a hyperbolic graph are in class $\mathcal{S}$ and we prove that locally compact wreath products $B \wr_X^A H$ are in class $\mathcal{S}$ whenever B is amenable, H is in class $\mathcal{S}$ and $H \curvearrowright X$ has amenable stabilizers (see Theorem 3.4.14 for a more precise statement). The suitable notion of wreath product for locally compact groups was introduced by Cornulier in [Cor17]. We recall this notion along with the notation used here in (3.4.9) on 130.

Easy examples of groups not belonging to class $\mathcal{S}$ are product groups $G \times H$ with G nonamenable and H noncompact (Proposition 3.4.6), nonamenable groups with noncompact center (Proposition 3.4.7) and nonamenable groups that are inner amenable at infinity (Proposition 3.4.8).

We will use class $\mathcal{S}$ in the next chapter to prove rigidity results for group measure space constructions by locally compact groups and group von Neumann algebras of locally compact groups.

Most of this chapter is based on the author's publication [Dep19]. The examples in Proposition C are based on the joint publication [BDV18] of the author with Arnaud Brothier and Stefaan Vaes.

Recall from the introduction that we assume all locally compact groups to be second countable in this thesis.

3.1 Definition and basic properties

As in Definition A from the introduction that we define class $\mathcal{S}$ for locally compact groups as follows. Recall from Definition 2.3.31 that a group is said to be exact if the operation of taking the reduced crossed product preserves exactness.

Definition 3.1.1. Let G be a locally compact group. We say that G has *property (S)* if there exists a $\|.\|$-continuous map $\eta : G \to \mathrm{Prob}(G)$ satisfying

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0 \tag{3.1.1}$$

uniformly on compact sets for $g, h \in G$.

We say that G belongs to *class* $\mathcal{S}$ if G is exact and has property (S).

It is worthwhile to note that there are currently no known examples of groups with property (S) that are not exact.

We have the following equivalent characterizations of property (S). Recall that we denote by $M(G)$ the space of all complex Radon measures on G, by $M(G)^+$ the space of positive Radon measures on G and by $\mathrm{Prob}(G)$ the space of Radon probability measures on G. We denote by $\mathcal{S}(G) = \left\{ f \in L^1(G)^+ \;\middle|\; \|f\|_1 = 1 \right\}$ the space of probability measures on G that are absolutely continuous with respect to the Haar measure. We equip $M(G)$ with the norm of total variation. There is an obvious G-equivariant norm preserving embedding $\mathcal{S}(G) \hookrightarrow \mathrm{Prob}(G)$.

Proposition 3.1.2. *Let G be a locally compact group. Then, the following are equivalent.*

(i) *G has property (S), i.e. there exists a $\|.\|$-continuous map $\eta : G \to \mathrm{Prob}(G)$ satisfying*

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

(ii) *There is a $\|.\|_1$-continuous map $\eta : G \to \mathcal{S}(G)$ satisfying*

$$\lim_{k\to\infty} \|\eta(gkh) - \lambda_g \eta(k)\|_1 = 0$$

uniformly on compact sets for $g, h \in G$.

(iii) *There is a $\|.\|_2$-continuous map $\xi : G \to L^2(G)$ with $\|\xi(g)\|_2 = 1$ for all $g \in G$ satisfying*

$$\lim_{k\to\infty} \|\xi(gkh) - \lambda_g \xi(k)\|_2 = 0$$

uniformly on compact sets for $g, h \in G$.

(iv) *There exists a Borel map $\eta : G \to \mathrm{Prob}(G)$ satisfying*

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

(v) *There exists a sequence of Borel maps $\eta_n : G \to M(G)^+$ satisfying*

$$\liminf_{n\to\infty} \liminf_{k\to\infty} \|\eta_n(k)\| > 0$$

and

$$\lim_{n\to\infty} \limsup_{k\to\infty} \sup_{g,h\in K} \|\eta_n(gkh) - g \cdot \eta_n(k)\| = 0$$

for all compact sets $K \subseteq G$.

Proof. First, we prove (i)⇔(ii). The implication from right to left is trivial. To prove the converse, we follow the lines of [Ana02, Proposition 2.2]. Let $\eta : G \to \mathrm{Prob}(G)$ be as in (i). We construct $\tilde{\eta} : G \to \mathcal{S}(G)$ as follows. Take an $f \in C_c(G)^+$ with $\int_G f(t)\,\mathrm{d}t = 1$. Define

$$\tilde{\eta}(g)(s) = \int_G f(t^{-1}s)\,\mathrm{d}\eta(g)(t)$$

for $s, g \in G$. One checks that $\tilde{\eta}(g) \in \mathcal{S}(G)$ for every $g \in G$ and that $\tilde{\eta}$ is $\|.\|_1$-continuous. For all $g, h, k \in G$ we have

$$\|\tilde{\eta}(gkh) - g \cdot \tilde{\eta}(k)\|_1 = \int_G |\tilde{\eta}(gkh)(s) - \tilde{\eta}(k)(g^{-1}s)|\,\mathrm{d}s$$

$$= \int_G \left| \int_G f(t^{-1}s) \, \mathrm{d}\eta(gkh)(t) - \int_G f(t^{-1}g^{-1}s) \, \mathrm{d}\eta(k)(t) \right| \mathrm{d}s$$

$$\leqslant \int_G \int_G f(t^{-1}s) \, \mathrm{d}|\eta(gkh) - g \cdot \eta(k)|(t) \, \mathrm{d}s$$

$$= \|\eta(gkh) - g \cdot \eta(k)\|$$

which tends to zero uniformly on compact sets for $g, h \in G$ whenever $k \to \infty$.

Next, we prove (ii)⇔(iii). Let $\eta : G \to \mathcal{S}(G)$ be a map as in (ii). Define the map $\xi : G \to L^2(G)$ by $\xi(g)(s) = \sqrt{\eta(g)(s)}$ for $g, s \in G$. Using that $|\sqrt{a} - \sqrt{b}|^2 \leqslant |a - b|$ for all $a, b \geqslant 0$, we get

$$\lim_{k\to\infty} \|\xi(gkh) - \lambda_g \xi(k)\|_2 \leqslant \lim_{k\to\infty} \|\eta(gkh) - \lambda_g \eta(k)\|_1 = 0,$$

uniformly on compact sets for $g, h \in G$. Conversely, let $\xi : G \to L^2(G)$ be a map as in (iii). Define $\eta : G \to \mathcal{S}(G)$ by $\eta(g)(s) = |\xi(g)(s)|^2$ for $g, s \in G$. Using Hölder's inequality and $\left||a|^2 - |b|^2\right| \leqslant \left||a| + |b|\right| \left|a - b\right|$ for $a, b \in \mathbb{R}$, we get

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\|_1 \leqslant 2 \lim_{k\to\infty} \|\xi(gkh) - \lambda_g \xi(k)\|_2 = 0,$$

uniformly on compact sets for $g, h \in G$.

The implications (i)⇒(iv)⇒(v) are trivial. We still have to prove the converse implications (v)⇒(iv)⇒(i).

The implication (iv)⇒(i) follows immediately by applying Lemma 3.1.5 below with $H = Y = G$ and the actions $G \times G \curvearrowright H$ and $G \curvearrowright Y$ defined by $(g, k) \cdot h = ghk^{-1}$ and $(g, h) \cdot y = gy$ for $g, k \in G$, $h \in H$ and $y \in Y$.

Finally, (v)⇒(iv) follows from the technical lemma 3.1.3 below applied on the spaces $X = Y = G$ with the same actions as above. □

The following is a more abstract and slightly more general version of the trick in [BO08, Exercise 15.1.1]. It will be used several times in this chapter.

Lemma 3.1.3. *Let X and Y be σ-compact, locally compact spaces and G a locally compact group. Suppose that $G \curvearrowright X$ and $G \curvearrowright Y$ are continuous actions. If there exists a sequence of Borel maps $\eta_n : X \to M(Y)^+$ satisfying*

$$\lim_{n\to\infty} \limsup_{x\to\infty} \sup_{g\in K} \|\eta_n(gx) - g \cdot \eta_n(x)\| = 0 \tag{3.1.2}$$

for all compact sets $K \subseteq G$ and

$$\liminf_{n\to\infty} \liminf_{x\to\infty} \|\eta_n(x)\| > 0.$$

Then, there exists a Borel map $\eta : X \to \mathrm{Prob}(Y)$ *such that*

$$\lim_{x\to\infty} \|\eta(gx) - g\cdot\eta(x)\| = 0 \tag{3.1.3}$$

uniformly on compact sets for $g \in G$. *Moreover, if the maps* η_n *are assumed to be* $\|.\|$*-continuous, then also* η *can be assumed to be* $\|.\|$*-continuous. If the maps* η_n *are weakly* continuous and* $x \mapsto \|\eta_n(x)\|$ *is continuous, then also* η *can be assume to be weakly* continuous.*

Proof. After passing to a subsequence and replacing values of η_n in a compact set, we can assume that there exists a $\delta > 0$ such that $\|\eta_n(x)\|_1 \geqslant \delta$ for all $n \in \mathbb{N}$ and all $x \in X$. Set $\tilde{\eta}_n(x) = \eta_n(x)/\|\eta_n(x)\|$ for all $x \in X$. Note that

$$\|\tilde{\eta}_n(gx) - g\cdot\tilde{\eta}_n(x)\| \leqslant \frac{2}{\|\eta_n(x)\|}\|\eta_n(gx) - g\cdot\eta_n(x)\|$$

for all $x \in X$, $g \in G$ and $n \in \mathbb{N}$. and hence the sequence $(\tilde{\eta}_n)_n$ still satisfies (3.1.2).

Take an increasing sequence $(K_n)_n$ of compact symmetric neighborhoods of the unit e in G such that $G = \bigcup_n \mathrm{int}(K_n)$. After passing to a subsequence of $(\tilde{\eta}_n)_n$, we find compact sets $L_n \subseteq X$ such that

$$\|\tilde{\eta}_n(gx) - g\cdot\tilde{\eta}_n(x)\| \leqslant 2^{-n+1}$$

for all $g \in K_n$ and $x \in X \setminus L_n$. After inductively enlarging L_n, we can assume that the sequence $(L_n)_n$ is increasing, that $gL_n \subseteq L_{n+1}$ for all $g \in K_n$ and that $X = \bigcup_n L_n$. Moreover, we can also assume that $L_0 = \emptyset$.

For every $x \in X$, we denote $|x| = \max\{n \in \mathbb{N} \mid x \notin L_n\}$. Furthermore, we denote $h(n) = \lfloor n/2 \rfloor + 1$ for $n \geqslant 1$. Fix a $y_0 \in Y$. We set $\mu(x) = \delta_{y_0}$ whenever $|x| = 0$ and

$$\tilde{\eta}(x) = \sum_{k=h(|x|)}^{|x|} \tilde{\eta}_k(x)$$

whenever $|x| \geqslant 1$. Now, define $\tilde{\eta} : X \to \mathrm{Prob}(Y)$ by $\eta(x) = \tilde{\eta}(x)/\|\tilde{\eta}(x)\|$.

We prove that η satisfies (3.1.3). Take $\varepsilon > 0$ and $K \subseteq G$ arbitrary. Since $\bigcup_n \mathrm{int}(K_n) = G$, we can take an $n_0 \geqslant 1$ such that $K \subseteq K_{n_0}$. Take $n_1 > \max\{2n_0, 16/\varepsilon\}$. We claim that $\|\eta(g\cdot x) - g\cdot\eta(x)\| < \varepsilon$ whenever $x \in G \setminus L_{n_1}$ and $g \in K$. Indeed, fix $g \in K$ and $x \in G \setminus L_{n_1}$. Take $n \geqslant n_1$ such that $x \in L_{n+1} \setminus L_n$. Then, $|x| = n$. Since $gL_{n+1} \subseteq L_{n+2}$ and $g^{-1}L_{n-1} \subseteq L_n$, we have that $gx \in L_{n+2} \setminus L_{n-1}$ and hence $n - 1 \leqslant |gx| \leqslant n + 1$. This yields

$$\|\tilde{\eta}(gx) - g\cdot\tilde{\eta}(x)\| \leqslant 2 + \sum_{k=h(n)}^{n} \|\tilde{\eta}_k(gx) - g\cdot\tilde{\eta}_k(x)\| \leqslant 4,$$

since $g \in K_k$ and $x \in X \setminus L_k$ for $k = h(n), \ldots, n$. Hence

$$\|\eta(gx) - g \cdot \eta(x)\| \leqslant \frac{2}{\|\tilde{\eta}(x)\|} \|\tilde{\eta}(gx) - g \cdot \tilde{\eta}(x)\| \leqslant \frac{4}{n} \cdot 4 < \varepsilon$$

which proves the claim.

If the maps η_n are $\|.\|$-continuous (resp. weakly* continuous and such that $x \mapsto \|\eta_n(x)\|$ is continuous), then we can make $\tilde{\eta}$ (and hence η) $\|.\|$-continuous (resp. weakly* continuous) in the following way. Note first that in that case the maps $\tilde{\eta}_n$ are also $\|.\|$-continuous (resp. weakly* continuous). By inductively enlarging the compact sets L_n above and using that X is locally compact, we can assume that $L_n \subseteq \mathrm{int}(L_{n+1})$. For all $n \geqslant 1$, we take a continuous function $f_n : X \to [0,1]$ such that $f_n(x) = 1$ if $x \in L_{2n} \setminus L_n$ and $f_n(x) = 0$ if $x \in L_{n-1}$ or $x \in X \setminus L_{2n+1}$. Take $f_0 : X \to [0,1]$ such that $f_0(x) = 1$ if $x \in L_1$ and $f_0(x) = 0$ if $x \in X \setminus L_2$. For $x \in X$, we set

$$\begin{aligned}\tilde{\eta}(x) &= \sum_{k=0}^{+\infty} f_k(x)\tilde{\eta}_k(x) \\ &= f_{h(|x|)-1}(x)\, \tilde{\eta}_{h(|x|)-1}(x) + f_{|x|+1}(x)\, \tilde{\eta}_{|x|+1}(x) + \sum_{k=h(|x|)}^{|x|} \tilde{\eta}_k(x).\end{aligned}$$

Again, we define $\eta : X \to \mathrm{Prob}(Y)$ by $\eta(x) = \tilde{\eta}(x)/\|\tilde{\eta}(x)\|$. Obviously, η is continuous and, by a similar calculation as above, one proves that η satisfies (3.1.3). □

Remark 3.1.4. Using almost exactly the same proof as above, one can actually prove the following slightly more general result: suppose that for every $\varepsilon > 0$, every compact set $K \subseteq G$, there exists a compact set $L \subseteq X$ such that for all compact sets $L' \subseteq X$, there exists a map $\eta' : X \to M(Y)^+$ such that

$$\frac{\|\eta'(gx) - g \cdot \eta'(x)\|}{\|\eta'(x)\|} < \varepsilon \tag{3.1.4}$$

whenever $g \in K$ and $x \in L' \setminus L$. Then, there exists a map $\eta : X \to \mathrm{Prob}(Y)$ as in (3.1.3). Indeed, using the notation of the proof, we can take the compact sets $L_n \subseteq X$ and the maps $\eta_n : X \to M(Y)^+$ such that

$$\|\tilde{\eta}_n(gx) - g \cdot \tilde{\eta}_n(x)\| \leqslant \frac{2}{\|\eta_n(x)\|} \|\eta_n(gx) - g \cdot \eta_n(x)\| < 2^{-n+1}$$

for all $g \in K_n$ and $x \in L_{2n} \setminus L_n$, where again $\tilde{\eta}_n(x) = \eta_n(x)/\|\eta_n(x)\|$. The rest of the proof holds verbatim.

The following lemma will be used several times to replace Borel maps by continuous maps.

Lemma 3.1.5. *Let H and G be locally compact groups and Y a locally compact space. Suppose that $G \curvearrowright^{\alpha} H$ and $G \curvearrowright Y$ are arbitrary continuous actions. If there exists a Borel map $\eta : H \to \mathrm{Prob}(Y)$ satisfying*

$$\lim_{h\to\infty} \big\|\eta\big(\alpha_g(h)\big) - g\cdot\eta(h)\big\| = 0 \quad \textit{and} \quad \lim_{h\to\infty} \big\|\eta\big(\alpha_g(h)k\big) - \eta\big(\alpha_g(hk)\big)\big\| = 0$$

uniformly on compact sets for $g \in G$ and $k \in H$, then there exists a $\|.\|$-continuous map $\tilde{\eta} : H \to \mathrm{Prob}(Y)$ map satisfying

$$\lim_{h\to\infty} \big\|\tilde{\eta}\big(\alpha_g(h)\big) - g\cdot\tilde{\eta}(h)\big\| = 0$$

uniformly on compact sets for $g \in G$.

Proof. Fix a compact neighborhood K of the unit e in H with $\lambda_H(K) = 1$. We define $\tilde{\eta} : H \to \mathrm{Prob}(Y)$ by

$$\tilde{\eta}(g) = \int_K \eta(gk)\,\mathrm{d}k.$$

The map $\tilde{\eta}$ is continuous, since for $h_1, h_2 \in H$ we have

$$\|\tilde{\eta}(h_1) - \tilde{\eta}(h_2)\| \leqslant \int_{h_1K\triangle h_2K} \|\eta(k)\|\,\mathrm{d}k = \lambda_H(h_1K\triangle h_2K)$$

and the right hand side tends to zero whenever $h_2 \to h_1$. Moreover, for $g \in G$ and $h \in H$, we have

$$\big\|\tilde{\eta}\big(\alpha_g(h)\big) - g\cdot\tilde{\eta}(h)\big\| \leqslant \int_K \big\|\eta\big(\alpha_g(h)k\big) - g\cdot\eta(hk)\big\|\,\mathrm{d}k$$

Since K is compact the right hand side tends to zero uniformly on compact sets for $g \in G$ whenever $h \to \infty$. □

We can now provide the first easy examples of locally compact groups in class $\mathcal{S}$.

Corollary 3.1.6. *Class $\mathcal{S}$ contains all amenable locally compact groups.*

Proof. Let G be amenable. Obviously, G is exact. Take a sequence $(\nu_n)_n$ in $\mathrm{Prob}(G)$ such that $\|\nu_n - g\cdot\nu_n\| \to 0$ uniformly on compact sets for $g \in G$ whenever $n \to \infty$. Then, the sequence of maps $\eta_n : G \to \mathrm{Prob}(G)$ defined by

$$\eta_n(g) = \nu_n$$

for all $g \in G$ and $n \in \mathbb{N}$ satisfies the conditions of Proposition 3.1.2 (v). □

We end this section by proving the following theorem. Recall from Section 2.3.1 that the left equivariant Stone-Čech compactification $\beta^{lu}G$ is defined as the spectrum of the C*-algebra $C_b^{lu}(G)$ of bounded left uniformly continuous functions $G \to \mathbb{C}$. We prove that if G is exact, then property (S) is equivalent with the existence of a map $\eta : G \to \mathrm{Prob}(\beta^{lu}G)$ satisfying (3.1.1). This was implicitly used before in [BO08, Chapter 15] for countable groups.

Theorem 3.1.7. *Let G be a locally compact group. Then G belongs to class $\mathcal{S}$ if and only if G is exact and there exists a Borel map $\eta : G \to \mathrm{Prob}(\beta^{lu}G)$ satisfying*

$$\lim_{k\to\infty} \|\eta(gkh) - g\cdot\eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

Proof. The implication from left to right is trivial. To prove the converse implication, note that by exactness of G and Lemma 2.1.19, the action $G \curvearrowright \mathrm{Prob}(\beta^{lu}G)$ is topologically amenable. Take a sequence $\theta_n : \mathrm{Prob}(\beta^{lu}G) \to \mathrm{Prob}(G)$ such that

$$\lim_{n\to\infty} \|\theta_n(g\cdot\mu) - g\cdot\theta_n(\mu)\| = 0$$

uniformly for $\mu \in \mathrm{Prob}(\beta^{lu}G)$ and uniformly on compact sets for $g \in G$. Now, for the composition $\eta_n = \theta_n \circ \eta$, we get

$$\begin{aligned}\|\eta_n(gkh) - g\cdot\eta_n(k)\| &\leqslant \|\eta(gkh) - g\cdot\eta(k)\| + \big\|\theta_n\big(g\cdot\eta(k)\big) - g\cdot\theta_n\big(\eta(k)\big)\big\| \\ &\leqslant \|\eta(gkh) - g\cdot\eta(k)\| + \sup_{\mu\in\mathrm{Prob}(\beta^{lu}G)} \big\|\theta_n(g\cdot\mu) - g\cdot\theta_n\big(\mu\big)\big\|\end{aligned}$$

whenever $g, h, k \in G$. It follows that $(\eta_n)_n$ satisfies the conditions of Proposition 3.1.2 (v). □

3.2 Class $\mathcal{S}$ and boundary actions small at infinity

Let G be a locally compact group. We say that a compactification $\overline{G}$ of G is *G-equivariant* if both the actions $G \curvearrowright G$ by left and by right translation extend to continuous actions $G \curvearrowright \overline{G}$. Following [BO08, Definition 15.2.4], one defined the following.

Definition 3.2.1. A G-equivariant compactification $\overline{G}$ of G is said to be *small at infinity* if the extension of the action by right translation is trivial on $\overline{G} \setminus G$.

We define the compactification h^uG as the spectrum of the C*-algebra

$$C(h^uG) \cong \{f \in C_b^u(G) \mid \rho_g f - f \in C_0(G) \text{ for all } g \in G\}$$

and denote by $\nu^u G = h^u G \setminus G$ its boundary. By definition, this compactification is G-equivariant and small at infinity. It moreover satisfies the following universal property.

Proposition 3.2.2. *Let G be a locally compact group. For every G-equivariant compactification $\overline{G}$ that is small at infinity, the inclusion $G \hookrightarrow \overline{G}$ extends to a continuous G-equivariant map $h^u G \to \overline{G}$ that is equivariant for the actions induced by left and right translation.*

Proof. Let $\overline{G}$ be a compactification that is small at infinity. For every $f \in C(\overline{G})$, we have that the restriction $h = f|_G$ is uniformly continuous and

$$\lim_{s \to \infty} |h(st) - h(s)| = 0$$

for every $t \in G$, since $\overline{G}$ is small at infinity. Hence, we can define the G-equivariant $*$-morphism

$$\pi : C(\overline{G}) \to C(h^u G) : f \mapsto f|_G.$$

Since π is the identity on $C_0(G)$, it follows that the continuous, G-equivariant map $h^u G \to \overline{G}$ induced by Gelfand duality extends the inclusion $G \hookrightarrow \overline{G}$. $\square$

We are now ready to prove Theorem B from the introduction.

Theorem 3.2.3. *Let G be a locally compact group. Then, the following are equivalent.*

- *(i)* *G is in class $\mathcal{S}$,*
- *(ii)* *the action $G \curvearrowright \nu^u G$ induced by left translation is topologically amenable,*
- *(iii)* *the action $G \curvearrowright h^u G$ induced by left translation is topologically amenable,*
- *(iv)* *the action of $G \times G$ on the spectrum of $C_b^u(G)/C_0(G)$ induced by left and right translation is topologically amenable.*

This characterization is a locally compact version of [Oza06, Theorem 4.1]. The two novelties in the proof of this result are the proof of (iii) and the method we used to prove the implication (iv)⇒(i). Indeed, in the original proof of Ozawa in the countable setting, it was proven that G belongs to class $\mathcal{S}$ if and only if G is exact and admits a u.c.p. map $\theta : \mathrm{C}^*_r(G) \otimes_{\min} \mathrm{C}^*_r(G) \to B\big(L^2(G)\big)$ such that $\theta(a \otimes b) - \lambda(a)\rho(b) \in K\big(L^2(G)\big)$. Here, λ and ρ denote the representations of $\mathrm{C}^*_r(G)$ induced by the left and right regular representation respectively. However, this characterization of class $\mathcal{S}$ does not hold for general locally compact groups. Indeed, for all connected groups G, the reduced group C^*-algebra $\mathrm{C}^*_r(G)$ is nuclear (see Theorem 2.3.29) and hence, using Theorem 2.3.26 and the Choi-Effros lifting theorem [CE76], one sees that a map θ as above always exists.

Proof of Theorem 3.2.3. First, we prove (i)⇒(ii). Let $\eta : G \to \mathrm{Prob}(G)$ be a map as in the definition of class $\mathcal{S}$. Consider the map $\eta_* : C_b^{lu}(G) \to C_b^{u}(G)$ defined by

$$(\eta_* f)(g) = \int_G f(s)\,\mathrm{d}\eta(g)(s)$$

for $f \in C_b^{lu}(G)$ and $g \in G$. Note that η_* is well-defined. Indeed, fix $f \in C_b^{lu}(G)$ and $\varepsilon > 0$. For all $g, h \in G$ we have

$$|(\eta_* f)(h^{-1}g) - (\eta_* f)(g)| \leqslant \|f\|_\infty \left\|\eta(h^{-1}g) - \eta(g)\right\|$$

and

$$|(\eta_* f)(h^{-1}g) - (\eta_* f)(g)| \leqslant \|f\|_\infty \left\|\eta(h^{-1}g) - h^{-1} \cdot \eta(g)\right\| + \|f - \lambda_h f\|_\infty .$$

Pick a compact neighborhood $K \subseteq G$ of identity. We find compact subset $L \subseteq G$ such that

$$\left\|\eta(h^{-1}g) - h^{-1} \cdot \eta(g)\right\| \leqslant \frac{\varepsilon}{2}$$

whenever $h \in K$ and $g \in G \setminus L$. Now, we can take an open neighborhood $\mathcal{U} \subseteq K$ of identity such that

$$\|f - \lambda_h f\| \leqslant \frac{\varepsilon}{2} \qquad \text{and} \qquad \sup_{g \in L} \left\|\eta(h^{-1}g) - \eta(g)\right\| \leqslant \varepsilon$$

for all $h \in \mathcal{U}$. It follows that

$$\|\lambda_h(\eta_* f) - \eta_* f\|_\infty \leqslant \varepsilon$$

for all $h \in \mathcal{U}$ and hence $\eta_* f$ is left uniformly continuous. Similarly, we have for $g, h \in G$

$$|(\eta_* f)(gh) - (\eta_* f)(g)| \leqslant \|f\|_\infty \|\eta(gh) - \eta(g)\| . \tag{3.2.1}$$

Picking again a compact neighborhood $K \subseteq G$ of identity, we find a compact subset $L \subseteq G$ such that

$$\|\eta(gh) - \eta(g)\| \leqslant \varepsilon$$

whenever $h \in K$ and $g \in G \setminus L$. Now, taking an open neighborhood $\mathcal{U} \subseteq K$ of identity such that

$$\sup_{g \in L} \|\eta(gh) - \eta(h)\| \leqslant \varepsilon$$

for all $h \in \mathcal{U}$, we conclude that

$$\|\rho_h(\eta_* f) - \eta_* f\|_\infty < \varepsilon$$

for all $h \in \mathcal{U}$ and hence that $\eta_* f$ is right uniformly continuous.

Moreover, (3.2.1) also implies that

$$\lim_{g \to \infty} |(\eta_* f)(gh) - (\eta_* f)(g)| = 0$$

for all $h \in G$ and hence that $\eta_*(f) \in C(h^u G)$ for all $f \in C_b^{lu}(G)$.

Similarly, one proves that $\eta_*(\lambda_g f) - \lambda_g(\eta_* f) \in C_0(G)$. Let $\pi : C(h^u G) \to C(\nu^u G) \cong C(h^u G)/C_0(G)$ be the quotient map. It follows that $\pi \circ \eta_* : C_b^{lu}(G) \to C(\nu^u G)$ is a G-equivariant. By dualization, we obtain a weakly* continuous G-equivariant map $\nu^u G \to \mathrm{Prob}(\beta^{lu} G)$ given by $x \mapsto \delta_x \mapsto \mu \circ \pi \circ \eta_*$. Since G is exact, the action $G \curvearrowright \beta^{lu} G$ is amenable and hence so is $G \curvearrowright \mathrm{Prob}(\beta^{lu} G)$ (see Lemma 2.1.19). It follows that $G \curvearrowright \nu^u G$ is amenable.

Now, we prove (ii)⇔(iii). The implication from right to left is trivial. To prove the other implication, take an arbitrary compact subset $K \subseteq G$ and an $\varepsilon > 0$. By Proposition 2.1.15, it suffices to construct a function $h \in C_c(h^u G \times G)^+$ such that $\int_G h(x,s)\,\mathrm{d}s = 1$ for every $x \in h^u G$ and

$$\int_G |h(x, g^{-1}s) - h(gx, s)|\,\mathrm{d}s < \varepsilon \tag{3.2.2}$$

for all $x \in h^u G$ and $g \in K$.

By Proposition 2.1.15 and Remark 2.1.17, we find an $f \in C_c(\nu^u G \times G)^+$ satisfying $\int_G f(x,s)\,\mathrm{d}s = 1$ and

$$\int_G |f(x, g^{-1}s) - f(gx, s)|\,\mathrm{d}s < \frac{\varepsilon}{2}$$

for all $x \in \nu^u G$ and $g \in K$. By the Tietze Extension Theorem, we can extend f to a function $\tilde{f} \in C_c(h^u G \times G)^+$. Since

$$\limsup_i \int_G |\tilde{f}(x_i, g^{-1}s) - \tilde{f}(gx_i, s)|\,\mathrm{d}s = \int_G |f(x, g^{-1}s) - f(gx, s)|\,\mathrm{d}s < \frac{\varepsilon}{2},$$

for every net $(x_i)_i$ in G converging to an $x \in \nu^u G$, we can take a compact set $L \subseteq G$ such that

$$\int_G |\tilde{f}(x, g^{-1}s) - \tilde{f}(gx, s)|\,\mathrm{d}s < \frac{\varepsilon}{2}.$$

for all $x \in h^u G \setminus L$ and $g \in K$. After possibly enlarging L and renormalizing $\tilde{f}$, we can moreover assume that $\int_G \tilde{f}(x,s)\,\mathrm{d}s = 1$.

Now, fix a function $a \in C_c(G)^+$ with $\int_G a(s)\,\mathrm{d}s = 1$. Using Lemma 3.2.4 below, we take a function $\zeta \in C_c(G)^+$ such that $\zeta|_L = 1$ and $|\zeta(gh) - \zeta(h)| < \varepsilon/4$ for $h \in G$ and $g \in K$. Now, define $h \in C_c(h^u G \times G)$ by

$$h(x,s) = \begin{cases} \zeta(x)a(x^{-1}s) + \big(1-\zeta(x)\big)\tilde{f}(x,s) & \text{if } x \in G, \\ \tilde{f}(x,s) & \text{if } x \in \nu^u G. \end{cases}$$

A straightforward calculation shows that h satisfies (3.2.2).

Next, we prove (ii)⇒(iv). Denote by X the spectrum of $A = C_b^u(G)/C_0(G)$. Since $C(h^u G) \subseteq C_b^u(G)$, we have a natural embedding $C(\nu^u G) \hookrightarrow A$, which in turn induces a continuous map $\varphi_\ell : X \to \nu^u G$. Note that φ_ℓ is $G\times G$-equivariant with respect to the actions induced by left and right translation. Similarly, we get a $G \times G$-equivariant map $\varphi_r : X \to \nu_r^u G$, where $\nu_r^u G$ denotes the spectrum of the algebra

$$C(\nu_r^u G) = \{f \in C_b^u(G) \mid \lambda_g f - f \in C_0(G)\}$$

and the action $G \times G \curvearrowright \nu_r^u G$ is induced by left and right translation. By assumption, the action $G \times 1 \curvearrowright \nu^u G$ is amenable, and by symmetry so is $1 \times G \curvearrowright \nu_r^u G$. Since the actions $1 \times G \curvearrowright \nu^u G$ and $G \times 1 \curvearrowright \nu_r^u G$ are trivial, the diagonal action $G \times G \curvearrowright \nu^u G \times \nu_r^u G$ is amenable. Now, the conclusion follows from the $G \times G$-equivariance of the map $\varphi_\ell \times \varphi_r : X \to \nu^u G \times \nu_r^u G$.

Finally, we prove (iv)⇒(i). By Theorem 2.3.34, the group G is exact. Denote again by X the spectrum of $A = C_b^u(G)/C_0(G)$. Denoting by $\beta^u G$ the spectrum of $C_b^u(G)$, we get $X = \beta^u G \setminus G$. By Proposition 2.1.15 and Remark 2.1.17, we can take a sequence $(f_n)_n$ of functions in $C_c(X \times G \times G)^+$ such that $\int_{G\times G} f_n(x,s,t)\,\mathrm{d}s\,\mathrm{d}t = 1$ for all $x \in X$ and $n \in \mathbb{N}$, and such that

$$\lim_{n\to\infty} \int_{G\times G} |f_n(x, g^{-1}s, h^{-1}t) - f_n\big((g,h)\cdot x, s, t\big)|\,\mathrm{d}s\,\mathrm{d}t = 0 \tag{3.2.3}$$

uniformly for $x \in X$ and uniformly on compact sets for $g, h \in G$. As before, the Tietze Extension Theorem yields extensions $\tilde{f}_n \in C_c(\beta^u G \times G \times G)^+$ of each f_n. For each $x \in \beta^u G$ and $n \in \mathbb{N}$, we define $\eta_n(x) \in M(G)^+$ as the measure with density function

$$s \mapsto \int_G \tilde{f}_n(x,s,t)\,\mathrm{d}t.$$

with respect to the Haar measure. This yields $\|.\|$-continuous maps $\eta_n : \beta^u G \to M(G)^+$. By (3.2.3), the restrictions of η_n to $G \subseteq \beta^u G$ satisfy the conditions of Proposition 3.1.2 (v). $\square$

In the proof above, we used the following easy lemma.

Lemma 3.2.4. *Let G be a locally compact group. For all compact subsets $K, L \subseteq G$ and all $\varepsilon > 0$, there exists a continuous function $f \in C_c(G)$ satisfying $f|_L = 1$ and*

$$|f(kgk') - f(g)| < \varepsilon$$

for $k, k' \in K$ and $g \in G$.

Proof. Without loss of generality we can assume that K is symmetric and that $\text{int}(K)$ contains the unit e. Denote $L_0 = L$ and $L_n = K^n L K^n$ for $n \geqslant 1$. Then, $L_n \subseteq \text{int}(L_{n+1})$ for every $n \in \mathbb{N}$ and hence, we can take a continuous $f_n : G \to [0,1]$ with $f_n(g) = 1$ for $g \in L_n$ and $\text{supp}\, f_n \subseteq L_{n+1}$. Take $N \in \mathbb{N}$ such that $1/N < \varepsilon/4$ and set

$$f(g) = \frac{1}{N} \sum_{k=0}^{N-1} f_k(g).$$

Then, f satisfies the conclusions of the lemma. □

3.3 Class $\mathcal{S}$ is a measure equivalence invariant

In this section, we prove Theorem E. For countable groups, this result was proven by Sako in [Sak09a].

Theorem 3.3.1. *The class $\mathcal{S}$ is closed under measure equivalence.*

As said before, by [DL15, Corollary 2.9] and [DL14, Theorem 0.1 (6)] exactness is preserved under measure equivalence. So, it suffices to prove that property (S) is a measure equivalence invariant. In order to prove that, we will use the characterization of measure equivalence in terms of cross section equivalence relations (see Theorem 2.6.4) and introduce a notion of property (S) for these relations.

Recall from Definition 2.5.3 the definition of a locally bounded set $\mathcal{W} \subseteq \mathcal{R}$ for a countable equivalence relations $\mathcal{R}$. Recall that

$$\mathcal{R}^{(2)} = \{(x, y, z) \in X \times X \times X \mid x \sim y \sim z\}.$$

We define property (S) for countable equivalence relations as follows.

Definition 3.3.2. Let $\mathcal{R}$ be a countable equivalence relation on a standard measure space (X, μ). We say that $\mathcal{R}$ has *property (S)* if there exists a Borel map $\eta : \mathcal{R}^{(2)} \to \mathbb{C}$ such that

$$\sum_{\substack{z \in X \\ z \sim x}} \eta(x, y, z) = 1$$

for a.e. $(x, y) \in \mathcal{R}$ and such that for all $\varepsilon > 0$ and $\varphi, \psi \in [\mathcal{R}]$ the set

$$\left\{(x, y) \in \mathcal{R} \;\middle|\; \sum_{\substack{z \in X \\ z \sim x}} |\eta\big(\varphi(x), \psi(y), z\big) - \eta(x, y, z)| \geqslant \varepsilon\right\} \tag{3.3.1}$$

is locally bounded.

Remark 3.3.3. The map η above can be viewed as a map assigning to all $(x, y) \in \mathcal{R}$ a probability measure on the orbit of y such that for all $\varepsilon > 0$ and all $\varphi, \psi \in [\mathcal{R}]$ the set

$$\big\{(x, y) \in \mathcal{R} \mid \big\|\eta\big(\varphi(x), \psi(y)\big) - \eta(x, y)\big\|_1 \geqslant \varepsilon\big\} \tag{3.3.2}$$

is locally bounded.

We prove first that this notion of property (S) is stable under restrictions and amplifications of ergodic, countable equivalence relation.

Lemma 3.3.4. *Let $\mathcal{R}$ be a countable, ergodic equivalence relation on some standard probability space (X, μ) and let $X_0 \subseteq X$ be a Borel subset with positive measure. Then, $\mathcal{R}$ has property (S) if and only if the restriction $\mathcal{R}_0 = \mathcal{R} \cap (X_0 \times X_0)$ has property (S).*

Proof. Since $\mathcal{R}$ is ergodic, we can take a partition $Y = \bigcup_i Y_i$ up to measure zero and Borel isometries $\varphi_i \in [\mathcal{R}]$ such that $\varphi_i(Y_i) \subseteq Y_0$.

Suppose first that $\mathcal{R}_0$ has property (S) and let η_0 be as in Definition 3.3.2. We extend η_0 to a map η on $\mathcal{R}$ by setting

$$\eta(x, y) = \eta_0\big(\varphi_i(x), \varphi_j(y)\big)$$

for every $(x, y) \in \mathcal{R}$ with $x \in Y_i$ and $y \in Y_j$. It is straightforward to check that η satisfies (3.3.1) for every $\varepsilon > 0$ and $\varphi, \psi \in [\mathcal{R}]$.

Conversely, suppose that $\mathcal{R}$ has property (S). Let η be a map as in the definition. Define for $(x, y) \in \mathcal{R}_0$ a probability measure on the $\mathcal{R}_0$-orbit of y by setting

$$\eta_0(x, y)(z) = \sum_{\substack{i \in I \\ z \in \varphi_i(Y_i)}} \eta(x, y)\big(\varphi_i^{-1}(z)\big)$$

whenever $(x, y) \in \mathcal{R}_0$. Clearly, η_0 satisfies (3.3.1) for every $\varepsilon > 0$ and every $\varphi, \psi \in [\mathcal{R}_0]$ □

It follows that property (S) is preserved under stable isomorphism.

Corollary 3.3.5. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two ergodic countable equivalence relations such that $\mathcal{R}_1 \cong_s \mathcal{R}_2$. Then, $\mathcal{R}_1$ has property (S) if and only $\mathcal{R}_2$ has.*

Recall from Section 2.5.5 that every action nonsingular, essentially free action $G \curvearrowright (X, \mu)$ admits a cross section $X_1 \subseteq X$ such that the cross section equivalence relation $\mathcal{R} = \mathcal{R}(G \curvearrowright X) \cap (X_1 \times X_1)$ is countable. We prove that the above notion of property (S) is compatible with taking cross section equivalence relations.

Proposition 3.3.6. *Let G be a locally compact group and $G \curvearrowright (X, \mu)$ an essentially free, ergodic, pmp action. Let $X_1 \subseteq X$ be a cross section and $\mathcal{R}$ the associated cross section equivalence relation. Then, G has property (S) if and only if $\mathcal{R}$ has property (S).*

Proof. Fix Borel maps $\gamma : X \to G$ and $\pi : X \to X_1$ such that $x = \gamma(x) \cdot \pi(x)$ for a.e. $x \in X$. First, assume that G has property (S). Let $\eta : G \to \mathrm{Prob}(G)$ be a map as in the definition of property (S). Define for each $x \in X$ a map

$$\pi_x : G \to X_1 : g \mapsto \pi(g^{-1}x).$$

Note that π_x is a Borel map from G to the $\mathcal{R}$-orbit of $\pi(x)$. We define the map η' as in Definition 3.3.2 by

$$\eta'(x, y) = (\pi_x)_* \eta\big(\omega(x, y)\big)$$

for $(x, y) \in \mathcal{R}$. Note that indeed every $\eta'(x, y)$ is a probability measure on the $\mathcal{R}$-orbit of x.

To prove that η' satisfies (3.3.2), fix $\varepsilon, \delta > 0$ and $\varphi, \psi \in [\mathcal{R}]$. It suffices to find a Borel set $E \subset X_1$ with $\mu_1(X_1 \setminus E) < \delta$ such that the set

$$\big\{(x, y) \in \mathcal{R} \cap (E \times E) \bigm| \big\|\eta'\big(\varphi(x), \psi(y)\big) - \eta'(x, y)\big\|_1 \geqslant \varepsilon\big\} \tag{3.3.3}$$

is bounded.

By Lemma 2.5.40, we find a compact set $K \subseteq G$ and a measurable $E \subseteq X_1$ with $\mu_1(X_1 \setminus E) < \delta$ such that $\omega\big(\varphi(x), x\big) \in K$ and $\omega\big(y, \psi(y)\big) \in K$ for all $x, y \in E$. Take a compact set $L \subset G$ such that $\|\eta(gkh) - g \cdot \eta(k)\|_1 < \varepsilon$ for all $g, h \in K$ and all $k \in G \setminus L$. We claim that

$$\big\|\eta'\big(\varphi(x), \psi(y)\big) - \eta'(x, y)\big\|_1 < \varepsilon \tag{3.3.4}$$

whenever $(x,y) \in \mathcal{R} \cap (E \times E)$ and $\omega(x,y) \in G \setminus L$. Assuming the claim is true, the set from (3.3.3) is contained in the set of all $(x,y) \in \mathcal{R}$ with $\omega(x,y) \in L$ which is bounded by Lemma 2.5.40. To prove (3.3.4), fix $(x,y) \in \mathcal{R} \cap (E \times E)$ with $\omega(x,y) \in G \setminus L$. We have

$$\left\|\eta'\big(\varphi(x), \varphi(y)\big) - \eta'(x,y)\right\|_1 = \left\|(\pi_{\varphi(x)})_*\eta\big(\omega(\varphi(x), \psi(y))\big) - (\pi_x)_*\eta\big(\omega(x,y)\big)\right\|_1$$

Now, $\pi_x(g) = \pi_{\varphi(x)}\big(\omega(\varphi(x), x)g\big)$ and hence

$$(\pi_x)_*\big(\eta\big(\omega(x,y)\big)\big) = (\pi_{\varphi(x)})_*\big(\omega\big(\varphi(x), x\big) \cdot \eta\big(\omega(x,y)\big)\big)$$

which yields that

$$\begin{aligned}\left\|\eta'\big(\varphi(x), \psi(y)\big) - \eta'(x,y)\right\|_1 &= \left\|\eta\big(\omega(\varphi(x), \psi(y))\big) - \omega\big(\varphi(x), x\big) \cdot \eta\big(\omega(x,y)\big)\right\|_1 \\ &< \varepsilon.\end{aligned}$$

where we used the identity

$$\omega\big(\varphi(x), \psi(x)\big) = \omega\big(\varphi(x), x\big)\omega(x,y)\omega\big(y, \psi(y)\big)$$

and the assumption that $\omega\big(\varphi(x), x\big), \omega\big(y, \psi(y)\big) \in K$ and $\omega(x,y) \in G \setminus L$. Hence, (3.3.4) is proved.

Conversely, assume that $\mathcal{R}$ has property (S) and let η be a map as in the definition. Choose an arbitrary $\xi \in \mathrm{Prob}(G)$ and define

$$\eta' : G \to \mathrm{Prob}(G) : g \mapsto \int_X \Big(\sum_{\substack{z \in X_1 \\ z \sim \pi(x)}} \eta\big(\pi(gx), \pi(x), z\big)\omega(gx, z) \cdot \xi \Big)\, \mathrm{d}\mu(x).$$

We prove that η' satisfies (3.1.1). To motivate the arbitrary choice of ξ, note that whenever η' satisfies (3.1.1), so does the map $g \mapsto \eta'(g) * \xi$, where $\eta'(g) * \xi$ denotes the convolution product of $\eta'(g), \xi \in \mathrm{Prob}(G)$.

Fix a symmetric, compact neighborhood K of the unit e in G and an $\varepsilon > 0$. Take a compact, symmetric subset $L \subseteq G$ such that $F = \gamma^{-1}(L)$ satisfies $\mu(F) \geqslant 1 - \varepsilon$. Denote $\kappa = \lambda_G(L)/\operatorname{covol}(X_1)$, where $\operatorname{covol}(X_1)$ is defined as in Remark 2.5.39. By Lemma 2.5.40, the set

$$\mathcal{W} = \{(x,y) \in \mathcal{R} \mid \omega(x,y) \in LKL\}$$

is bounded Borel. Writing $\mathcal{W}$ as a union of finitely many elements of $[[\mathcal{R}]]$ and using (3.3.2), we see that the set

$$\mathcal{V} = \{(x,y) \in \mathcal{R} \mid \exists (x,x'), (y,y') \in \mathcal{W}, \|\eta(x',y') - \eta(x,y)\|_1 \geqslant \varepsilon\}$$

is locally bounded. Denoting $\delta = \varepsilon/\kappa$ and using Lemma 2.5.40, we can find a compact set $C \subseteq G$ and a measurable $E \subseteq X_1$ and with $\mu_1(E) \geqslant 1 - \delta$ such that $\omega\big(\mathcal{V} \cap (E \times E)\big) \subseteq C$. We conclude that

$$\|\eta(x', y') - \eta(x, y)\|_1 < \varepsilon \tag{3.3.5}$$

whenever $(x, y) \in \mathcal{R} \cap (E \times E)$ with $(x, x') \in \mathcal{W}$, $(y, y') \in \mathcal{W}$ and $\omega(x, y) \in G \setminus C$.

Denote $D = LCL$. We conclude the proposition by proving that

$$\|\eta'(gkh) - g \cdot \eta'(k)\| < 4\kappa\delta + 9\varepsilon = 13\varepsilon \tag{3.3.6}$$

for all $g, h \in K$ and $k \in G \setminus D$. So, fix $g, h \in K$ and $k \in G \setminus D$. Applying the change of variables $x \mapsto h^{-1}x$ and using that $\omega(gkx, z) = g\omega(kx, z)$, we find that

$$\eta'(gkh) = g \cdot \left(\int_X \Big(\sum_{\substack{z \in X_1 \\ z \sim \pi(x)}} \eta\big(\pi(gkx), \pi(h^{-1}x), z\big) \omega(kx, z) \cdot \xi \Big) \, \mathrm{d}\mu(x) \right)$$

and hence

$$\|\eta'(gkh) - g \cdot \eta'(k)\| \leqslant \int_X \big\|\eta\big(\pi(gkx), \pi(h^{-1}x)\big) - \eta\big(\pi(kx), \pi(x)\big)\big\|_1 \, \mathrm{d}\mu(x).$$

Since $g, h^{-1} \in K$, we have that $\big(\pi(gkx), \pi(kx)\big) \in \mathcal{W}$ and $\big(\pi(h^{-1}x), \pi(x)\big) \in \mathcal{W}$ whenever $x \in X$ is such that $gkx, h^{-1}x, kx, x \in F = \gamma^{-1}(L)$. Moreover, for such an x we also have $\omega\big(\pi(kx), \pi(x)\big) \in LkL \subseteq G \setminus C$. Hence, by (3.3.5) we have that

$$\big\|\eta\big(\pi(gkx), \pi(h^{-1}x)\big) - \eta\big(\pi(kx), \pi(x)\big)\big\|_1 < \varepsilon \tag{3.3.7}$$

whenever $gkx, h^{-1}x, kx, x \in F$, $\pi(x) \in E$ and $\pi(kx) \in E$.

Since $\mu(F) \geqslant 1 - \varepsilon$, we can find a measurable set F' with $\mu(F') \geqslant 1 - 4\varepsilon$ such that $gkx, h^{-1}x, kx, x \in F$ for every $x \in F'$. Moreover, the map $\theta : G \times X_1 \to X$ is injective on the image A of the map $x \mapsto \big(\gamma(x), \pi(x)\big)$. Hence by Proposition 2.5.38 and Remark 2.5.39, we have that $\operatorname{covol}(X_1)\, \mu\big(\theta(\mathcal{U})\big) = (\lambda_G \otimes \mu_1)(\mathcal{U})$ for all $\mathcal{U} \subseteq A$. It follows that for measurable $S \subseteq X_1$, we have that

$$\begin{aligned} \mu\big(\pi^{-1}(S) \cap F\big) &= \operatorname{covol}(X_1)^{-1} (\lambda_G \times \mu_1)\big(A \cap (L \times S)\big) \\ &\leqslant \frac{\lambda_G(L)}{\operatorname{covol}(X_1)} \mu_1(S) = \kappa\mu_1(S). \end{aligned}$$

Applying this to $\pi^{-1}(X_1 \setminus E) \cap F$ and using the definition F' above, we conclude that (3.3.7) holds on a set whose complement has at most measure $4\varepsilon + 2\kappa\delta$ and hence that (3.3.6) holds. $\square$

We now have the following.

Corollary 3.3.7. *Property (S) is a measure equivalence invariant.*

Proof. Let G be a locally compact group in class $\mathcal{S}$ and let H be a group that is measure equivalent to G. By [KKR18, Theorem A] and [KKR17, Theorem A], G and H admit free, ergodic, probability measure preserving actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ with cross sections $X_1 \subseteq X$, $Y_1 \subseteq Y$ and cross section equivalence relations $\mathcal{R}$ and $\mathcal{T}$ respectively such that $\mathcal{R}$ is stably isomorphic to $\mathcal{T}$. But, by Proposition 3.3.6, the relation $\mathcal{R}$ (resp. $\mathcal{T}$) has property (S) if and only if G (resp. H) has and by Corollary 3.3.5, $\mathcal{R}$ has property (S) if and only if $\mathcal{T}$ has. □

The proof of Theorem 3.3.1 is now immediate.

Proof of Theorem 3.3.1. Let G be a locally compact group in class $\mathcal{S}$ and let H be a group that is measure equivalent to G. Then, H has property (S) by the previous result. Moreover, by [DL15, Corollary 2.9] and [BCL17, Theorem A] G is exact if and only if the proper metric space (G, d) has property (A) in the sense of Roe, where d is any proper left invariant metric that implements the topology on G, and by [DL14, Theorem 0.1 (6)], property A is a measure equivalence invariant. □

In [Oza09], Ozawa proved that $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is in class $\mathcal{S}$. Combining this with Theorem 3.3.1 yields the following.

Corollary 3.3.8. *The groups $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ both belong to class $\mathcal{S}$.*

Proof. The group $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a lattice in both $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. Hence, the latter two groups are measure equivalent with $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$, which belongs to class $\mathcal{S}$ by [Oza09]. □

3.4 Examples of groups of class $\mathcal{S}$

As said in the introduction, several examples of countable, discrete groups in class $\mathcal{S}$ were already known: amenable groups, hyperbolic groups (see [Ada94]), lattices in connected simple Lie groups of real rank one with finite center (see [Ska88, Théorème 4.4]), wreath products $B \wr \Gamma$ with B amenable and Γ in class $\mathcal{S}$ and $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ (see [Oza09]) are all in class $\mathcal{S}$.

In this section, we provide several examples of locally compact groups in class $\mathcal{S}$. Recall from Corollary 3.1.6 and Corollary 3.3.8 that class $\mathcal{S}$ contains all amenable groups, $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. By [Ska88, Proof of Théorème 4.4], the following holds.

Proposition 3.4.1. *Connected, simple Lie groups of real rank one with finite center belong to class $\mathcal{S}$.*

As mentioned in Example 2.1.9, examples of connected, simple Lie groups of real rank one with finite center are $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SU}(n,1)$ and $\mathrm{Sp}(n,1)$, along with the connected component of $\mathrm{SO}(n,1)$ containing the unit element.

Skandalis obtained the previous result by using that connected, simple Lie groups of real rank one with finite center have negative sectional curvature. Below, we will obtain a similar result for other "negatively curved" groups: we prove below that groups acting continuously and properly on a tree or a hyperbolic graph are also in class $\mathcal{S}$. This proves for instance that the totally disconnected group $\mathrm{SL}_2(K)$ with K a local field (for instance the field $\mathbb{Q}_p$ of p-adic numbers) belongs to class $\mathcal{S}$ (see for instance [Ser80, Chapter II] for the construction of the action of $\mathrm{SL}_2(K)$ on a tree). We prove furthermore that certain locally compact wreath products $B \wr_X^A H$ (see Theorem 3.4.14) are in class $\mathcal{S}$. Along the way, we characterize when semidirect products belongs to class $\mathcal{S}$ (see Proposition 3.4.11). Finally, we also provide some counterexamples to groups in class $\mathcal{S}$.

We begin by proving the following result which states that we can ignore closed, amenable subgroups when proving that a group belongs to class $\mathcal{S}$.

Proposition 3.4.2. *Let G be an locally compact group and K a closed, amenable subgroup. If there exists a Borel map $\eta : G \to \mathrm{Prob}(G/K)$ such that*

$$\lim_{k\to\infty} \|\eta(gkh) - g\cdot\eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$. Then, G has property (S).

Proof. Using Lemma 3.1.5 we can assume that η is $\|.\|$-continuous. The proof then follows easily from Lemma 3.4.4 below. □

Let G be a locally compact group and $H \subseteq G$ a closed subgroup. Denote by $p : G \to G/H$ the quotient map. Let $\sigma : G/H \to G$ be a locally bounded Borel section for p, i.e. a Borel map satisfying $p \circ \sigma = \mathrm{id}_{G/H}$ that maps compact sets onto precompact sets (see for instance [Mac52, Lemma 1.1] for the existence of such a map). We can identify G with $G/H \times H$ via the map

$$\phi : G \to G/H \times H : g \mapsto \big(gH, \sigma(gH)^{-1}g\big). \tag{3.4.1}$$

Under this identification the action by left translation is given by $k \cdot (gH, h) = \big(kgH, \omega(k, gH)h\big)$, where $\omega(k, gH) = \sigma(kgH)^{-1}k\sigma(gH)$. Note that ω maps compact sets of $G \times G/H$ onto precompact sets of G.

The identification map ϕ is not continuous, but it is bimeasurable and maps (pre)compact sets to precompact sets. This allows us to identify the spaces $\mathrm{Prob}(G)$ and $\mathrm{Prob}(G/H \times H)$ via the map $\mu \mapsto \phi_*\mu$. Note that this identification map is continuous with respect to the norm topology on both spaces (and hence bimeasurable), but not with respect to the weak* topology on both spaces. We use the above identifications in the following two lemmas.

Lemma 3.4.3. *Let G be a locally compact group and $H \subseteq G$ a closed, amenable subgroup. Let $(\nu_n)_n$ be a sequence in* $\mathrm{Prob}(H)$ *satisfying*

$$\lim_{n\to\infty} \|h \cdot \nu_n - \nu_n\| = 0$$

uniformly on compact sets for $h \in H$. Then,

$$\lim_{n\to\infty} \|g \cdot (\mu \otimes \nu_n) - (g \cdot \mu) \otimes \nu_n\| = 0$$

uniformly on compact sets for $g \in G$ and $\mu \in$ $\mathrm{Prob}(G/H)$, where we equipped $\mathrm{Prob}(G/H)$ *with the weak* topology.*

Proof. Fix compact subsets $K \subseteq G$ and $\mathcal{L} \subseteq \mathrm{Prob}(G/H)$. Take an arbitrary $\varepsilon > 0$. There is a compact subset $L \subseteq G/H$ such that $\mu(L) > 1 - \varepsilon$ for all $\mu \in \mathcal{L}$. Hence, for all $f \in C_c(G/H \times H)$, $\mu \in \mathcal{L}$, $k \in G$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\left| \int_{G/H\times H} f \,\mathrm{d}k \cdot (\mu \otimes \nu_n) - \int_{G/H\times H} f \,\mathrm{d}(k \cdot \mu) \otimes \nu_n \right| \\
&\quad\leqslant \int_{G/H} \left| \int_H f\big(k \cdot (gH, h)\big) \,\mathrm{d}\nu_n(h) - \int_H f(kgH, h) \,\mathrm{d}\nu_n(h) \right| \mathrm{d}\mu(gH) \\
&\quad= \int_{G/H} \left| \int_H f\big(kgH, \omega(k, gH)h\big) \,\mathrm{d}\nu_n(h) - \int_H f(kgH, h) \,\mathrm{d}\nu_n(h) \right| \mathrm{d}\mu(gH) \\
&\quad\leqslant \int_{G/H} \int_H |f(kgH, h)| \,\mathrm{d}\big|\omega(k, gH) \cdot \nu_n - \nu_n\big|(h) \,\mathrm{d}\mu(gH) \\
&\quad\leqslant \|f\|_\infty \left(2\varepsilon + \int_L \|\omega(k, gH) \cdot \nu_n - \nu_n\| \,\mathrm{d}\mu(gH) \right),
\end{aligned}$$

where $\big|\omega(k, gH) \cdot \nu_n - \nu_n\big|$ denotes the total variation measure of $\omega(k, gH) \cdot \nu_n - \nu_n$. Since ω maps compact sets to precompact sets, we can find an $n_0 \in \mathbb{N}$ such that $\|\omega(k, gH) \cdot \nu_n - \nu_n\| < \varepsilon$ for all $n \geqslant n_0$, all $k \in K$ and all $gH \in L$. We conclude that

$$\|k \cdot (\mu \otimes \nu_n) - (k \cdot \mu) \otimes \nu_n\| \leqslant 3\varepsilon$$

whenever $n \geqslant n_0$, $\mu \in \mathcal{L}$ and $k \in K$, thus proving the result. ☐

Lemma 3.4.4. *Let G and H be locally compact groups, $\pi : G \to H$ a continuous morphism and $K \subseteq H$ a closed, amenable subgroup. Let $G \curvearrowright X$ be a continuous action on some σ-compact space X. Let $G \curvearrowright \mathrm{Prob}(H)$ (resp. $G \curvearrowright \mathrm{Prob}(H/K)$) be defined by $g \cdot \mu = \pi(g) \cdot \mu$ for $g \in G$ and $\mu \in \mathrm{Prob}(H)$ (resp. $\mu \in \mathrm{Prob}(H/K)$). If there exists a weakly* continuous map $\eta : X \to \mathrm{Prob}(H/K)$ such that*

$$\lim_{x \to \infty} \|\eta(gx) - g \cdot \eta(x)\| = 0$$

uniformly on compact sets for $g \in G$. Then, there exists a Borel map $\tilde{\eta} : X \to \mathrm{Prob}(H)$ such that

$$\lim_{x \to \infty} \|\tilde{\eta}(gx) - g \cdot \tilde{\eta}(x)\| = 0$$

uniformly on compact sets for $g \in G$. Moreover, if η is assumed to be $\|.\|$-continuous then also $\tilde{\eta}$ can be assumed to be $\|.\|$-continuous.

Proof. Fixing a locally bounded Borel section $\sigma : H/K \to H$ for the quotient map $p : H \to H/K$, we can identify H with $H/K \times K$ and $\mathrm{Prob}(H)$ with $\mathrm{Prob}(H/K \times K)$ as in (3.4.1).

Since K is amenable, we can take a sequence $(\nu_n)_n$ in $\mathrm{Prob}(K)$ such that $\|k \cdot \nu_n - \nu_n\| \to 0$ uniformly on compact sets for $k \in K$ whenever $n \to \infty$. We construct maps as in Remark 3.1.4 as follows. Fix an $\varepsilon > 0$ and a compact $C \subseteq G$. Take a compact $L \subseteq X$ such that $\|\eta(gx) - g \cdot \eta(x)\| < \varepsilon$ for all $g \in C$ and $x \in X \setminus L$. Fix any compact set $L' \subseteq X$. Applying Lemma 3.4.3 to the weak* compact set $\eta(L')$, we find an $n \in \mathbb{N}$ such that

$$\big\| \big(g \cdot \eta(x)\big) \otimes \nu_n - g \cdot \big(\eta(x) \otimes \nu_n\big) \big\| < \varepsilon$$

for any $x \in L'$ and $g \in C$. Defining $\eta : X \to \mathrm{Prob}(H)$ by $\eta'(x) = \eta(x) \otimes \nu_n$ for $x \in X$, we have

$$\begin{aligned} \|\eta'(gx) - g \cdot \eta'(x)\| &\leqslant \|\eta(gx) - g \cdot \eta(x)\| + \big\| \big(g \cdot \eta(x)\big) \otimes \nu_n - g \cdot \big(\eta(x) \otimes \nu_n\big) \big\| \\ &\leqslant 2\varepsilon \end{aligned}$$

for any $g \in C$ and any $x \in L' \setminus L$. We conclude that the map η' is as in (3.1.4). Moreover, if η is $\|.\|$-continuous, then so is η'. ☐

Class $\mathcal{S}$ is closed under passing to closed subgroups.

Proposition 3.4.5. *Let G be a locally compact group and $H \subseteq G$ a closed subgroup. If G is in class $\mathcal{S}$, then so is H.*

Proof. By [KW99a, Theorem 4.1], the subgroup H is exact. Fix a locally bounded Borel section $\sigma : G/H \to G$ for the quotient map $p : G \to G/H$ satisfying $\sigma(H) = e$. Identifying $\operatorname{Prob}(G)$ with $\operatorname{Prob}(G/H \times H)$ as in (3.4.1), we find an H-equivariant isometry $\operatorname{Prob}(G) \to \operatorname{Prob}(H)$. Composing the map $\eta : G \to \operatorname{Prob}(G)$ from the definition of property (S) for G with this isometry and restricting to H, yields the required map $H \to \operatorname{Prob}(H)$. □

We will now provide a few easy examples of groups not belonging to class $\mathcal{S}$.

Proposition 3.4.6. *Let G and H be two locally compact groups. Suppose that G is nonamenable and H is not compact. Then, the direct product $G \times H$ does not belong to class $\mathcal{S}$*

Proof. Suppose that $G \times H$ does belong to class $\mathcal{S}$. Let $\eta : G \times H \to \operatorname{Prob}(G \times H)$ be a map as in the definition. Since H is not compact, we can take a sequence $h_n \in H$ such that $h_n \to \infty$. We have

$$\lim_{n\to\infty} \|\eta(e, h_n) - g \cdot \eta(e, h_n)\| = \lim_{n\to\infty} \left\|\eta(gg^{-1}, h_n) - g \cdot \eta(e, h_n)\right\| = 0$$

uniformly on compact sets for $g \in G$. Integrating the measures $\eta(e, h_n)$ over H yields a sequence $(\nu_n)_n$ in $\operatorname{Prob}(G)$ such that $\|\nu_n - g \cdot \nu_n\| \to \infty$ uniformly on compact sets for $g \in G$. By Theorem 2.1.2 (vi) this contradicts the nonamenability of G. □

Together with Proposition 3.4.5, the previous result implies that $\mathrm{SL}(n, \mathbb{R})$ for $n \geqslant 3$ does not belong to class $\mathcal{S}$. Indeed, $\mathrm{SL}(n, \mathbb{R})$ contains the closed subgroup $\mathrm{SL}(n-1, \mathbb{R}) \times \mathbb{R}$, which does not belong to class $\mathcal{S}$ by the previous result.

Similarly, we prove the following.

Proposition 3.4.7. *Nonamenable locally compact groups with noncompact center do not belong to class $\mathcal{S}$.*

Proof. Suppose that G belongs to class $\mathcal{S}$. Let $\eta : G \to \operatorname{Prob}(G)$ be a sequence as in the definition of property (S). Take a sequence $(z_n)_n$ in the center $\mathcal{Z}(G)$ such that $z_n \to \infty$. Then,

$$\lim_{n\to\infty} \|\eta(z_n) - g \cdot \eta(z_n)\| = \lim_{n\to\infty} \left\|\eta(g z_n g^{-1}) - g \cdot \eta(z_n)\right\| = 0$$

uniformly on compact sets for $g \in G$. By Theorem 2.1.2 (vi), we conclude that G is amenable. □

Recall from Definition 2.1.11 that a group G is said to be inner amenable at infinity if G admits a conjugation invariant mean $m : L^\infty(G) \to \mathbb{C}$.

Proposition 3.4.8. *A locally compact group G that is inner amenable at infinity belongs to class $\mathcal{S}$ if and only if it is amenable.*

Proof. If G is amenable, the result is immediate. Conversely, suppose that G is in class $\mathcal{S}$. Let $\eta : G \to \mathrm{Prob}(G)$ be a map as in the definition. Define the map $\eta_* : C_b(G) \to C_b(G)$ by

$$(\eta_* f)(g) = \int_G f \, \mathrm{d}\eta(g).$$

It is easy to prove that $\eta_*(\lambda_g f) - \lambda_g \eta_*(f) \in C_0(G)$ and $\eta_*(f) - \rho_g \eta_*(f) \in C_0(G)$ for all $f \in C_b(G)$ and $g \in G$.

Since G is inner amenable at infinity, we can take a state $m : L^\infty(G) \to \mathbb{C}$ that is invariant under conjugation and such that $m(f) = 0$ for all $f \in C_0(G)$. Hence,

$$m \circ \eta_*(\lambda_g f) = m\big(\lambda_g \eta_*(f)\big) = m\big(\rho_g \lambda_g \eta_*(f)\big) = m \circ \eta_*(f)$$

for $f \in C_b(G)$. We conclude that $m \circ \eta_*$ is a left invariant mean on $C_b(G)$. □

3.4.1 Groups acting on trees or hyperbolic graphs

Recall from Section 2.2 the definition of a hyperbolic metric space. In this section, we prove Proposition C from the introduction. In the case of countable discrete groups, this result was proved by [Ada94].

Proposition 3.4.9. *Let G be a locally compact group. If one of the two following conditions is satisfied, then G belongs to class $\mathcal{S}$.*

(i) *G admits a continuous action on a (not necessarily locally finite) tree that is metrically proper in the sense that for every vertex x, we have that $d(x, g \cdot x) \to \infty$ when g tends to infinity in G.*

(ii) *G admits a continuous, proper action on a hyperbolic graph with uniformly bounded degree.*

Proof. In both cases, G is exact by Propositions 2.1.7 and 2.2.4 and Theorem 2.3.33. In both cases, we have an action $G \curvearrowright \mathcal{G}$ on a graph $\mathcal{G} = (V, E)$ that is metrically proper and continuous. We claim that it suffices to construct a map $\eta : G \to \mathrm{Prob}(V)$ satisfying

$$\lim_{k \to \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$. Indeed, suppose that η is such a map. Fix a maximal set of vertices $F \subseteq V$ with pairwise disjoint orbits and write $M_x = \lambda_G\big(\mathrm{Stab}(x)\big)$ for every $x \in V$. Define the G-equivariant map $\theta : \mathrm{Prob}(V) \to \mathcal{S}(G)$ by

$$\theta(\mu)(g) = \sum_{x \in F} \frac{1}{M_x} \mu(gx). \tag{3.4.2}$$

Note that the sum on the right hand side has at most countably many nonzero terms, since $\mu(Gx) > 0$ for at most countably $x \in F$. Using that $GF = V$, one easily checks that indeed $\theta(\mu) \in \mathcal{S}(G)$. Moreover, for $\mu, \nu \in \mathrm{Prob}(G)$, we have

$$\begin{aligned}\|\theta(\mu) - \theta(\nu)\|_1 &\leqslant \int_G \sum_{x \in F} \frac{1}{M_x} |\mu(gx) - \nu(gx)| \,\mathrm{d}g \\ &= \sum_{x \in F} \sum_{g \in G/S_x} |\mu(gx) - \nu(gx)| \\ &= \|\mu - \nu\|.\end{aligned}$$

Hence, the composition $\theta \circ \eta$ is as in the definition of property (S). This proves the claim.

Assume that $\mathcal{G}$ is tree. For all $x, y \in V$, denote by $[x, y] \subset V$ the (unique) geodesic between x and y. Fix a base point $x_0 \in V$. Define the continuous map $\eta : G \to \mathrm{Prob}(V)$ by defining $\eta(g)$ as the uniform probability measure on $[x_0, gx_0]$. For all $g, h, k \in G$ the symmetric difference between $[x_0, ghkx_0]$ and $g \cdot [x_0, hx_0] = [gx_0, ghx_0]$ is contained in $[x_0, gx_0] \cup [ghx_0, ghkx_0]$ and hence contains at most $d(x_0, gx_0) + d(x_0, kx_0)$ elements. Since the action $G \curvearrowright \mathcal{G}$ is metrically proper, we have $d(x_0, hx_0) \to \infty$ when h tends to infinity in G. It then follows that

$$\lim_{h \to \infty} \|\eta(ghk) - g \cdot \eta(h)\| \leqslant \lim_{h \to \infty} \frac{2}{d(x_0, hx_0)} \big(d(x_0, gx_0) + d(x_0, kx_0)\big) = 0$$

uniformly on compact sets of $g, k \in G$, as was required.

Now, assume that $G \curvearrowright \mathcal{G}$ is a continuous, proper action on a hyperbolic graph $\mathcal{G} = (V, E)$. This proof is based on [Kai04, Theorem 1.33] and especially the version in [BO08, Theorem 5.3.15]. For completeness, we provide the details here.

We use the following ad hoc terminology. Assume that $[x', y'] \subset V$ is a geodesic. If $d(x', y')$ is even, we call the "midpoint of $[x', y']$" the unique point $z \in [x', y']$ with $d(x', z) = d(z, y') = d(x', y')/2$. If $d(x', y')$ is odd, we declare two points of

$[x', y']$ to be the "midpoints of $[x', y']$", namely the two points $z \in [x', y']$ with $d(x', z) = (d(x', y') \pm 1)/2$ and thus $d(z, y') = (d(x', y') \mp 1)/2$. For all $x, y \in V$ and $k \in \mathbb{N}$, define $A(x, y, k) \subset V$ as the subset of all midpoints of all geodesics $[x', y'] \subseteq V$ with $d(x, x') \leqslant k$ and $d(y, y') \leqslant k$. Note that $A(x, y, k) = A(y, x, k)$ and $A(g \cdot x, g \cdot y, k) = g \cdot A(x, y, k)$ for all $x, y \in V$, $k \in \mathbb{N}$ and $g \in G$.

Take $\delta > 0$ such that every geodesic triangle in $\mathcal{G}$ is δ-thin (see [BO08, Definition 5.3.3]). Define

$$B = \sup_{x \in V} |\{y \in V \mid d(y, x) \leqslant 2\delta\}|.$$

Since $\mathcal{G}$ has uniformly bounded degree, we have that $B < \infty$. We claim that for all $x, y \in V$ with $d(x, y) \geqslant 4k$, we have

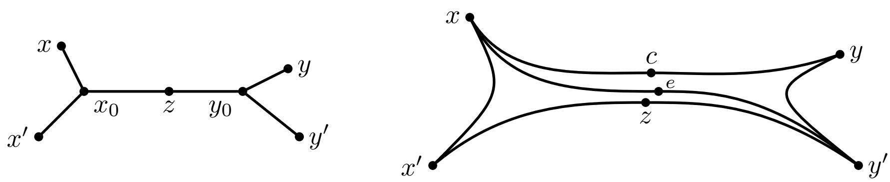
$$|A(x, y, k)| \leqslant 2(k + 1)B. \tag{3.4.3}$$

To prove this claim, fix a geodesic $[x, y]$ between x and y and denote by $[a, b] \subset [x, y]$ the unique segment determined by

$$d(x, a) = \lfloor d(x, y)/2 \rfloor - k \quad \text{and} \quad d(x, b) = \lceil d(x, y)/2 \rceil + k\ .$$

Note that $[a, b]$ contains at most $2(k + 1)$ vertices. To prove the claim, it thus suffices to show that every $z \in A(x, y, k)$ lies at distance at most 2δ from a vertex on $[a, b]$.

Choose a geodesic $[x', y'] \subset V$ with $d(x, x') \leqslant k$ and $d(y, y') \leqslant k$. Let z be one of the mid points of $[x', y']$. Since $d(x, y) \geqslant 4k$, the geodesic picture of the five points x, x', y, y', z in a *tree* would look as the following picture on the left.



In our comparison tree, some of the "small" segments $[x, x_0]$, $[x', x_0]$, $[y, y_0]$, $[y', y_0]$ could be reduced to a single point, but the "large" segment $[x_0, y_0]$ has length at least $2k$. Therefore, in the comparison tree, the mid point z of $[x', y']$ lies on the segment $[x_0, y_0]$. We now turn back to segments in the hyperbolic graph $\mathcal{G}$, as in the picture on the right. Denote by $c \in [x, y]$ the unique point with $d(x, c) = d(x', z)$. By construction, $c \in [a, b]$. To conclude the proof of (3.4.3), we show that $d(c, z) \leqslant 2\delta$. Choose a geodesic $[x, y']$ and denote by $e \in [x, y']$ the unique point with $d(x, e) = d(x', z)$. Applying δ-thinness to the geodesic triangle x, x', y', we find that $d(z, e) \leqslant \delta$. Then applying δ-thinness to the geodesic triangle x, y, y', we get that $d(e, c) \leqslant \delta$. So, $d(z, c) \leqslant 2\delta$ and the claim in (3.4.3) is proven.

Given a finite subset $A \subset V$, denote by $p(A)$ the uniform probability measure on A. Exactly as in the proof of [BO08, Theorem 5.3.15], define the sequence of maps

$$\eta_n : V \times V \to \mathrm{Prob}(V) : (x,y) \mapsto \frac{1}{n} \sum_{k=n+1}^{2n} p(A(x,y,k)) \; .$$

For finite sets $A, B \subset V$, we have

$$\begin{aligned} \|p(A) - p(B)\| &= |A \cap B| \left(\frac{1}{m} - \frac{1}{M} \right) + \frac{1}{|A|} |A \setminus B| + \frac{1}{|B|} |B \setminus A| \\ &= 2 - 2 \frac{|A \cap B|}{M}, \end{aligned}$$

where $M = \max\{|A|, |B|\}$ and $m = \min\{|A|, |B|\}$. When $d(x,x') \leqslant d \leqslant k$, we have

$$A(x,y,k-d) \subset A(x',y,k) \subset A(x,y,k+d).$$

Therefore, whenever $d(x,x') \leqslant d \leqslant k$, we have

$$\frac{1}{2} \|p(A(x,y,k)) - p(A(x',y,k))\| \leqslant 1 - \frac{|A(x,y,k-d)|}{|A(x,y,k+d)|} \; .$$

So if $d(x,x') \leqslant d \leqslant n$, we use the inequality between arithmetic and geometric mean and get that

$$\begin{aligned} \frac{1}{2} \|\eta_n(x,y) - \eta_n(x',y)\| &\leqslant 1 - \frac{1}{n} \sum_{k=n+1}^{2n} \frac{|A(x,y,k-d)|}{|A(x,y,k+d)|} \\ &\leqslant 1 - \left(\prod_{k=n+1}^{2n} \frac{|A(x,y,k-d)|}{|A(x,y,k+d)|} \right)^{1/n} \\ &= 1 - \left(\frac{\prod_{k=n-d+1}^{n+d} |A(x,y,k)|}{\prod_{k=2n-d+1}^{2n+d} |A(x,y,k)|} \right)^{1/n} \\ &\leqslant 1 - \frac{1}{|A(x,y,2n+d)|^{2d/n}}. \end{aligned}$$

Using (3.4.3), it follows that whenever $d(x,x') \leqslant d \leqslant n$ and $d(x,y) \geqslant 4(2n+d)$, we have

$$\frac{1}{2} \|\eta_n(x,y) - \eta_n(x',y)\| \leqslant 1 - \big(2(2n+d+1)B\big)^{-2d/n} \; .$$

So, for every $d \in \mathbb{N}$ and every $x \in V$, we have

$$\lim_{n\to\infty} \limsup_{y\to\infty} \sup_{\substack{x'\in V \\ d(x,x')\leqslant d}} \|\eta_n(x,y) - \eta_n(x',y)\| = 0$$

Now, fix a vertex $x_0 \in V$ and define maps $\tilde{\eta}_n : G \to \mathrm{Prob}(V)$ by $\tilde{\eta}_n(g) = \eta_n(x_0, gx_0)$ for every $g \in G$ and every $n \in \mathbb{N}$. For all $g, k \in G$ and $n \in \mathbb{N}$, we have

$$\|\tilde{\eta}_n(gk) - g \cdot \tilde{\eta}_n(k)\| = \|\eta_n(x_0, gkx_0) - \eta_n(gx_0, gkx_0)\|$$

and since the action $G \curvearrowright \mathcal{G}$ is continuous and proper, it follows that

$$\lim_{n\to\infty} \limsup_{k\to\infty} \sup_{g\in K} \|\tilde{\eta}_n(gk) - g \cdot \tilde{\eta}(k)\| = 0.$$

On the other hand, for all $k, h \in G$ and $n \in \mathbb{N}$, we have

$$\tilde{\eta}_n(kh) = \eta_n(x_0, khx_0) = k \cdot \eta_n(k^{-1}x_0, hx_0) = k \cdot \eta_n(hx_0, k^{-1}x_0)$$

and

$$\tilde{\eta}_n(k) = k \cdot \eta_n(k^{-1}x_0, x_0) = k \cdot \eta_n(x_0, k^{-1}x_0).$$

Hence

$$\|\tilde{\eta}_n(kh) - \tilde{\eta}_n(k)\| = \left\|\eta_n(hx_0, k^{-1}x_0) - \eta_n(x_0, k^{-1}x_0)\right\|.$$

It follows that

$$\lim_{n\to\infty} \limsup_{k\to\infty} \sup_{h\in K} \|\tilde{\eta}_n(kh) - \tilde{\eta}(k)\| = 0.$$

We conclude that the composition of η_n with the map θ from (3.4.2) yields maps as in Proposition 3.1.2 (v). □

Corollary 3.4.10. *All compactly generated hyperbolic groups G belong to class $\mathcal{S}$.*

Proof. By [CCMT15, Corollary 2.6], G admits a proper, continuous, cocompact, isometric action on a proper geodesic hyperbolic metric space. By [MMS04, Theorem 21 and Proposition 8], G satisfies at least one of the following three structural properties: G is amenable, or G admits a proper action on a hyperbolic graph with uniformly bounded degree, or G admits closed subgroups $K < G_0 < G$ such that G_0 is of finite index and open in G, K is a compact normal subgroup of G_0 and G_0/K is a real rank one, connected, simple Lie group with finite center. In the first two cases, G belongs to class $\mathcal{S}$ by Corollary 3.1.6 and Proposition 3.4.9. In the latter case G_0/K belongs to class $\mathcal{S}$ by Proposition 3.4.1. But, G_0/K is measure equivalent to G and hence the result follows from Theorem 3.3.1. □

3.4.2 Semidirect products and wreath products in class $\mathcal{S}$

The next result characterizes when a semidirect product belongs to class $\mathcal{S}$. By definition a semidirect product $G = B \rtimes H$ belongs to class $\mathcal{S}$ whenever it is exact and there exists a map $\eta : G \to \mathrm{Prob}(G)$ satisfying

$$\left\|\mu\big((a,k)(b,h)(a',k')\big) - (a,k)\cdot\mu(b,h)\right\| \to 0$$

uniformly on compact sets for $(a,k),(a',k') \in G$ whenever $(b,h) \to \infty$. The result below shows that is suffices that there exist *two* such maps one of which satisfies the convergence above when $b \to \infty$ and the other when $h \to \infty$.

Proposition 3.4.11. *Let $G = B \rtimes_\alpha H$ be a semidirect product of locally compact groups. Then, G is in class $\mathcal{S}$ if and only if B and H are exact, and there exists Borel maps $\mu : G \to \mathrm{Prob}(G)$ and $\nu : G \to \mathrm{Prob}(G)$ such that*

$$\lim_{b\to\infty} \left\|\mu\big((a,k)(b,h)(a',k')\big) - (a,k)\cdot\mu(b,h)\right\| = 0 \tag{3.4.4}$$

uniformly on compact sets for $a, a' \in B$ and $k, h, k' \in H$, and such that

$$\lim_{h\to\infty} \left\|\nu\big((a,k)(b,h)(a',k')\big) - (a,k)\cdot\nu(b,k)\right\| \tag{3.4.5}$$

uniformly for $b \in B$ and uniformly on compact sets for $a, a' \in B$ and $k, k' \in H$.

Proof. The only if part is immediate. Indeed, the map $\eta : G \to \mathrm{Prob}(G)$ as in the definition of class $\mathcal{S}$, satisfies both (3.4.4) and (3.4.5). Moreover, the groups B and H are exact as closed subgroups of an exact group (see [KW99a, Theorem 4.1]).

To prove the converse, note first that G is exact as an extension of an exact group by an exact group (see [KW99a, Theorem 5.1]). Let $\mu : G \to \mathrm{Prob}(G)$ and $\nu : G \to \mathrm{Prob}(G)$ be as above. By Proposition 3.1.2 (v), it suffices to prove that for every compact $K \subseteq G$ and every $\varepsilon > 0$, there exists a Borel map $\eta : G \to \mathrm{Prob}(G)$ and a compact $L \subseteq G$ such that

$$\left\|\eta\big((a,k)(b,h)(a',k')\big) - (a,k)\cdot\eta(b,h)\right\| < \varepsilon \tag{3.4.6}$$

for all $(a,k),(a',k') \in K$ and all $(b,h) \in G \setminus L$.

So, fix a compact $K \subseteq G$ and an $\varepsilon > 0$. Let $K_B \subseteq B$ and $K_H \subseteq H$ be compact subsets such that $K \subseteq \{(b,h) \mid b \in K_B, h \in K_H\}$. By assumption, we can take a compact set $\widetilde{L}_H \subseteq H$ such that

$$\left\|\nu\big((a,k)(b,h)(a',k')\big) - (a,k)\cdot\nu(b,k)\right\| < \frac{\varepsilon}{2} \tag{3.4.7}$$

whenever $a, a' \in K_B$, $b \in B$, $k, k' \in K_H$ and $h \in H \setminus \widetilde{L}_H$.

Using Lemma 3.2.4, we take a function $f \in C_c(H)$ such that $f(h) = 1$ for $h \in \widetilde{L}_H$ and $|f(khk') - f(h)| < \varepsilon/4$ whenever $h \in H$ and $k, k' \in K_H$. Set $L_H = \operatorname{supp} f$. Now, we can take a compact set $L_B \subseteq B$ such that

$$\left\|\mu\big((a,k)(b,h)(a',k')\big) - (a,k)\cdot\mu(b,h)\right\| < \frac{\varepsilon}{2} \tag{3.4.8}$$

whenever $a, a' \in K_B$, $b \in G \setminus L_B$, $k, k' \in K_H$ and $h \in L_H$.

Define $\eta : G \to \operatorname{Prob}(G)$ by

$$\eta(b,h) = f(h)\mu(b,h) + \big(1 - f(h)\big)\nu(b,h)$$

for $(b,h) \in G$. Set $L = \{(b,h) \in G \mid b \in L_B, h \in L_H\}$. Fix $(a,k), (a',k') \in K$ and $(b,k) \in G \setminus L$. Denote $g = (b,h)$, $g' = (a,k)(b,h)(a',k')$. We have

$$\begin{aligned}\|\eta(g') - (a,k)\cdot\eta(g)\| &\leqslant f(h)\,\|\mu(g') - (a,k)\cdot\eta(g)\| \\ &\qquad + \big(1 - f(h)\big)\,\|\nu(g') - (a,k)\cdot\nu(g)\| + 2|f(h) - f(khk')| \\ &\leqslant f(h)\,\|\mu(g') - (a,k)\cdot\eta(g)\| \\ &\qquad + \big(1 - f(h)\big)\,\|\nu(g') - (a,k)\cdot\nu(g)\| + \frac{\varepsilon}{2}\end{aligned}$$

We are in one of the following three cases (see Fig. 3.1).

Case 1. If $h \in H \setminus L_H$, then $f(h) = 0$ and (3.4.7) holds.

Case 2. If $h \in L_H \setminus \widetilde{L}_H$ and $b \in B \setminus L_B$, then both (3.4.7) and (3.4.8) hold.

Case 3. If $h \in \widetilde{L}_H$ and $b \in B \setminus L_B$, then $f(h) = 1$ and (3.4.8) holds.

In all three cases, we conclude that (3.4.6) holds, thus proving the proposition. □

Remark 3.4.12. Note that (3.4.4) is equivalent with the existence of a map $\tilde{\mu} : B \to \operatorname{Prob}(G)$ satisfying

$$\lim_{b\to\infty} \|\tilde{\mu}(aba') - a\cdot\tilde{\mu}(b)\| = 0 \qquad \text{and} \qquad \lim_{b\to\infty} \left\|\tilde{\mu}\big(\alpha_h(b)\big) - h\cdot\tilde{\mu}(b)\right\| = 0$$

uniformly on compact sets for $a, a' \in B$ and $h \in H$. Indeed, the restriction of a map as in (3.4.4) satisfies the above equations. Conversely, given a map $\tilde{\mu}$ as above, the map $\mu : G \to \operatorname{Prob}(G)$ defined by $\mu(b,h) = \tilde{\mu}(b)$ satisfies (3.4.4).

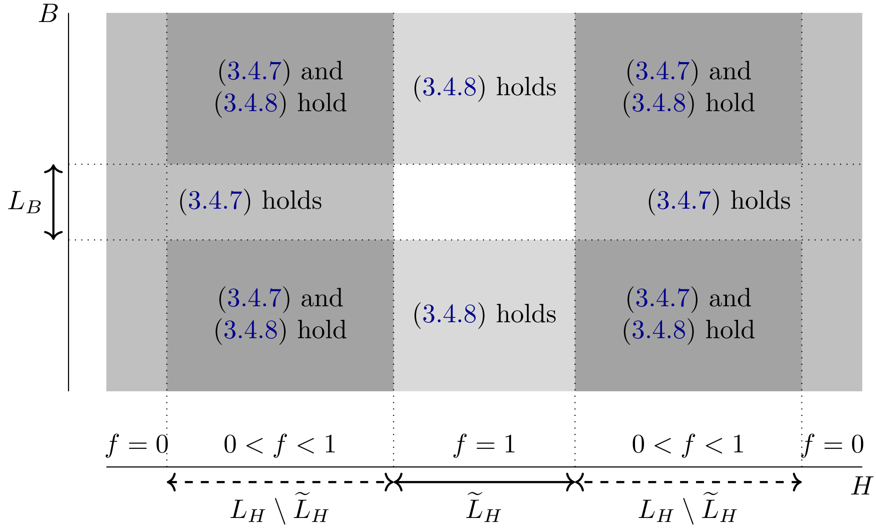


Figure 3.1: Regions where (3.4.7) and (3.4.8) hold.

When the group B is amenable, the previous result specializes to the corollary below. In the setting of countable groups, this result was proved by Ozawa in [Oza06, proof of Corollary 4.5] and [Oza09, Section 3]. However, the proof provided there does not carry over to the locally compact setting, since, as we explained after Theorem 3.2.3, the characterization of class $\mathcal{S}$ in terms of a u.c.p. map $\varphi : \mathrm{C}^*_r(G) \otimes_{\min} \mathrm{C}^*_r(G) \to B(L^2(G))$ satisfying $\varphi(x \otimes y) - \lambda(x)\rho(y) \in K(L^2(G))$ (see [BO08, Proposition 15.1.4]) does not hold in this setting. Also the method used in [BO08, Section 15.2] cannot be applied, since for a locally compact group G the crossed product $C(X) \rtimes_r G$ can be nuclear while $G \curvearrowright X$ is not amenable.

Corollary 3.4.13. *Let $G = B \rtimes_\alpha H$ be a semidirect product of locally compact groups with B amenable. Then G is in class $\mathcal{S}$ if and only if H is in class $\mathcal{S}$ and there is a Borel map $\mu : B \to \mathrm{Prob}(H)$ such that*

$$\lim_{b\to\infty} \left\|\mu\big(\alpha_h(b)\big) - h \cdot \mu(b)\right\| = 0 \qquad \textit{and} \qquad \lim_{b\to\infty} \|\mu(aba') - \mu(b)\| = 0$$

uniformly on compact sets for $h \in H$ and $a, a' \in B$.

Proof. The only if part follows immediately from Proposition 3.4.11 and Remark 3.4.12. Conversely, let $\mu : B \to \mathrm{Prob}(H)$ be a map as above. By Lemma 3.1.5, we can assume that μ is $\|.\|$-continuous. Let $H \curvearrowright^\beta B \times B$ be the diagonal action and let $G_0 = (B \times B) \rtimes_\beta H$. We apply Lemma 3.4.4

to the map μ with the morphism $\pi : G_0 \to G$ given by $\pi\big(a, a', h\big) = (a, h)$ for $a, a' \in B$ and $h \in H$, the space $X = B$, the action $G_0 \curvearrowright X$ given by $\big(a, a', h\big) \cdot b = a\alpha_h(b)(a')^{-1}$ for $a, a', b \in B$ and $h \in H$, and the amenable subgroup $K = B \subseteq G$. This yields a map $\tilde{\mu} : B \to \text{Prob}(G)$ as in Remark 3.4.12. Similarly, let $\eta : H \to \text{Prob}(H)$ be as in the definition of class $\mathcal{S}$. Applying Lemma 3.4.4 to the map ν with the morphism $\pi : G \times G \to G$ given by $\pi(g, g') = g$ for $g, g' \in G$, the space $X = H$, the action $G \times G \curvearrowright X$ induced by left and right translation and the amenable subgroup $K = B$, yields a map $\tilde{\nu} : H \to \text{Prob}(G)$ satisfying

$$\lim_{h\to\infty} \|\tilde{\nu}(khk') - (a, k) \cdot \tilde{\nu}(h)\| = 0$$

uniformly on compact sets for $k, k' \in G$ and $a \in B$. Now, the map $\nu : G \to \text{Prob}(G)$ defined by $\nu(b, h) = \tilde{\nu}(h)$ satisfies (3.4.5). □

We are now ready to characterize when wreath products belong to class $\mathcal{S}$. The suitable notion of wreath products for locally compact groups was introduced by Cornulier in [Cor17]. Let B and H be locally compact groups, X a countable set with continuous action $H \curvearrowright X$ and $A \subseteq B$ a compact open subgroup. The *semirestricted power* $B^{X,A}$ is defined by

$$B^{X,A} = \left\{(b_x)_{x\in X} \in B^X \;\middle|\; b_x \in A \text{ for all but finitely many } x \in X\right\}.$$

This space is locally compact and second countable when equipped with the topology generated by the open sets $\prod_{x\in X} C_x$ where $C_x \subseteq B$ is open for every $x \in X$ and $C_x = A$ for all but finitely many $x \in X$. For $b \in B^{X,A}$, we denote $\text{supp}_A b = \{x \in X \mid b(x) \notin A\}$.

Denote by α the action of H on $B^{X,A}$ by translation, i.e. $\alpha_h(b)(x) = b(h^{-1}x)$ for $b \in B^{X,A}$, $h \in H$ and $x \in X$. It is easy to see that this action is continuous. The *(semirestricted) wreath product* $B\wr^A_X$ is now defined as

$$B \wr^A_X H = B^{X,A} \rtimes_\alpha H \tag{3.4.9}$$

equipped with the product topology. By [Cor17, Proposition 2.4] it is a locally compact group.

Theorem D from the introduction is now an immediate consequence of the following theorem.

Theorem 3.4.14. *Let A, B, X and H be as above. Suppose that B is noncompact and $|X| \geqslant 2$. Then, $B \wr^A_X H$ belongs to class $\mathcal{S}$ if and only if B is amenable, the stabilizer* $\text{Stab}_H(x)$ *of every point $x \in X$ is amenable and H belongs to class $\mathcal{S}$.*

Note that if $|X| = 1$, then $B \wr_X^A H \cong B \times H$ belongs to class $\mathcal{S}$ if and only if both factors are amenable, or one of B and H belongs to class $\mathcal{S}$ and the other is compact. If B is compact, then by Theorem 3.3.1, we have that $B \wr_X^A H$ belongs to class $\mathcal{S}$ if and only if H does.

Proof of Theorem 3.4.14. Suppose first that $B \wr_X^A H$ belongs to class $\mathcal{S}$. It follows that the subgroups H and $B \times B$ do. Hence, B must be amenable. For every point $x_0 \in X$ the subgroup

$$B \times \mathrm{Stab}_H(x_0) \cong \big\{(b,h) \in B \wr_X^A G \;\big|\; b(x) = e \text{ if } x \neq x_0\big\}$$

belongs to class $\mathcal{S}$. Since B is noncompact, it follows that $\mathrm{Stab}_H(x_0)$ is amenable.

Conversely, suppose that H belongs to class $\mathcal{S}$ and that B and all stabilizers $\mathrm{Stab}_H(x)$ are amenable. We prove that $B \wr_X^A H$ belongs to class $\mathcal{S}$. Denote by $X = \bigcup_{i \in I} X_i$ the partition of X into the orbits of $H \curvearrowright X$ and fix $x_i \in X_i$ for all $i \in I$. Write $B_i = B^{X_i, A}$ and $H_i = \mathrm{Stab}_H(x_i)$.

Step 1. Each $B \wr_{X_i}^A H$ belongs to class $\mathcal{S}$. Fix $i \in I$. To prove this step, we proceed along the lines of [BO08, Corollary 15.3.6]. We claim that it suffices to prove the existence of a continuous map $\zeta_i : B_i \to M(H/H_i)^+ \cong \ell_1(X_i)^+$ satisfying

$$\lim_{b \to \infty} \frac{\big\|h \cdot \zeta_i(b) - \zeta_i\big(\alpha_h(b)\big)\big\|_1}{\|\zeta_i(b)\|_1} = 0 \qquad \text{and} \qquad \lim_{b \to \infty} \frac{\|\zeta_i(aba') - \zeta_i(b)\|_1}{\|\zeta_i(b)\|_1} = 0 \tag{3.4.10}$$

uniformly on compact sets for $h \in H$ and $a, a' \in B_i$. Indeed, suppose that ζ_i is such a map. Let $H \curvearrowright^{\beta} B_i \times B_i$ be the diagonal action. We first normalize ζ_i and then apply Lemma 3.4.4 to this normalized map with the groups $G = (B_i \times B_i) \rtimes H$, H and $K = H_i$, the space $X = B_i$, the morphism $\pi : G \to H$ given by $\pi(a, a', h) = h$, and the action $G \curvearrowright X$ given by $(a, a', h) \cdot b = a\alpha_h(b)(a')^{-1}$ for $a, a', b \in B_i$ and $h \in H$. This yields a map $\tilde{\zeta}_i : B_i \to \mathrm{Prob}(H)$ that satisfies the conditions of Corollary 3.4.13.

By [Str74] every locally compact group G admits a continuous proper length function, i.e. a continuous function $\ell : G \to \mathbb{R}^+$ such that $\ell(gh) \leqslant \ell(g) + \ell(h)$ and $\ell(g) = \ell(g^{-1})$ for all $g, h \in G$ and such that all the sets $\{g \in G \mid \ell(g) \leqslant M\}$ for $M > 0$ are compact. Fix such continuous, proper length functions $\ell_B : B \to \mathbb{R}^+$ and $\ell_H : H \to \mathbb{R}^+$. Define the function

$$f : X_i \to \mathbb{R}^+ : x \mapsto \inf_{\substack{h \in H \\ hx_i = x}} \ell_H(h).$$

Note that for every $M > 0$ the set $\{x \in X_i \mid f(x) \leqslant M\}$ is finite and that $f(hx) \leqslant \ell_H(h) + f(x)$. Similarly, we define

$$g : B \to \mathbb{R}^+ : b \mapsto \inf_{a,a' \in A} \ell_B(aba'),$$

and note that for every $M > 0$ the set $\{b \in B \mid g(b) \leqslant M\}$ is compact and that $g(bb') \leqslant g(b) + g(b') + N$, where $N = \sup_{a\in A} \ell_B(a)$. Also note that, by compactness of A, the map g is continuous.

Define $\zeta_i : B_i \to \ell^1(X_i)^+$ by

$$\zeta_i(b)(x) = \begin{cases} g\big(b(x)\big) + f(x) & \text{if } x \in \operatorname{supp}_A(b), \\ 0 & \text{otherwise} \end{cases}$$

for $b \in B_i$ and $x \in X_i$.

We prove that ζ_i satisfies (3.4.10). Fix $h \in H$ and $a, a', b \in B_i$. Denote $b' = aba'$, $S = \operatorname{supp}_A b$, $S' = \operatorname{supp}_A b'$ and $T = \operatorname{supp}_A a \cup \operatorname{supp}_A a'$. We have

$$\big\|h \cdot \zeta(b) - \zeta\big(\alpha_h(b)\big)\big\|_1 = \sum_{x\in hS} |f(h^{-1}x) - f(x)| \leqslant |S|\, \ell_H(h)$$

and

$$\begin{aligned} \|\zeta(b') - \zeta(b)\|_1 &= \sum_{x\in T} |\zeta(b')(x) - \zeta(b)(x)| \\ &= \sum_{x\in T\cap S\cap S'} \big|g\big(b'(x)\big) - g\big(b(x)\big)\big| + \sum_{x\in (T\cap S)\setminus S'} |g\big(b(x)\big) + f(x)| \\ &\quad + \sum_{x\in (T\cap S')\setminus S} |g\big(b'(x)\big) + f(x)| \\ &\leqslant \sum_{x\in T} \big|g\big(b'(x)\big) - g\big(b(x)\big)\big| + \sum_{x\in T\cap(S\triangle S')} f(x) \\ &\leqslant \sum_{x\in T} \Big(g\big(a'(x)\big) + g\big(a(x)\big) + 2N\Big) + \sum_{x\in T\cap(S\triangle S')} f(x) \\ &\leqslant \|\zeta(a)\|_1 + \|\zeta(a')\|_1 + 2N\,|T| \end{aligned}$$

where we used in the third step that $g(b) = 0$ whenever $b \in A$.

So, it suffices to prove that

$$\lim_{b\to\infty} \|\zeta_i(b)\|_1 = +\infty \qquad \text{and} \qquad \lim_{b\to\infty} \frac{|\operatorname{supp}_A b|}{\|\zeta_i(b)\|_1} = 0.$$

To prove the first, suppose that $\|\zeta(b)\|_1 \leqslant M$ for some $M > 0$. Then, $f(x) \leqslant M$ and $g\big(b(x)\big) \leqslant M$ for every $x \in \operatorname{supp}_A(b)$. Hence,

$$b \in C = \prod_{x \in X_i} C_x$$

where $C_x = \{b \in B \mid g(b) \leqslant M\}$ for $x \in F = \{x \in X \mid f(x) \leqslant M\}$ and $C_x = A$ otherwise. Since F is finite and each C_x is compact, it follows that C is compact, which in turn implies the claim.

To prove the other assertion, suppose that $|\operatorname{supp}_A b|/\|\zeta_i(b)\|_1 \geqslant \delta$ for some $b \in B$ and $\delta > 0$. Denote $D = \{x \in X_i \mid f(x) \leqslant 2/\delta\}$. Then,

$$\frac{2}{\delta}\big(|\operatorname{supp}_A b| - |D|\big) \leqslant \frac{2}{\delta}|\operatorname{supp}_A b \setminus D| \leqslant \|\zeta_i(b)\|_1 \leqslant \frac{1}{\delta}|\operatorname{supp}_A b|$$

and thus $|\operatorname{supp}_A b| \leqslant 2|D|$. It follows that $\|\zeta_i(b)\|_1 \leqslant \frac{2}{\delta}|D|$. But, by the previous, the set

$$\left\{b \in B \;\middle|\; \|\zeta(b)\|_1 \leqslant \frac{2}{\delta}|D|\right\}$$

is compact and hence so is $\{b \in B \mid |\operatorname{supp}_A b|/\|\zeta_i(b)\|_1 \geqslant \delta\}$.

Step 2. Construction of maps $\xi_i : B_i \to \operatorname{Prob}(H)$ satisfying (3.4.11) below. Fix $i \in I$, $\varepsilon > 0$ and a compact $K \subseteq H$. In this step, we construct a Borel map $\xi_i : B_i \to \operatorname{Prob}(H)$ such that

$$\big\|\xi_i\big(\alpha_h(b)\big) - h \cdot \xi_i(b)\big\| \leqslant \varepsilon \qquad \text{and} \qquad \xi_i(aba') = \xi_i(b) \tag{3.4.11}$$

for all $b \in B_i \setminus A^{X_i}$, all $h \in K$ and all $a, a' \in A^{X_i}$. Note that the difference with the previous step is that we want the map ξ_i to satisfy (3.4.11) for all $b \in B_i \setminus A^{X_i}$, instead of $b \in B_i \setminus L$ for L some (possibly large) compact set.

Since H_i is amenable, the action $H \curvearrowright H/H_i$ is topologically amenable. Indeed, let $(\nu_n)_n$ be a sequence in $\operatorname{Prob}(H_i)$ such that $\|h \cdot \nu_n - \nu_n\| \to 0$ uniformly on compact sets for $h \in H_i$ when $n \to \infty$. Fix a section $\sigma : H/H_i \to H$ for the quotient map $p : H \to H/H_i$. Then, the sequence of maps $\eta_n : H/H_i \to \operatorname{Prob}(H)$ defined by

$$\eta_n(hH) = \sigma(hH) \cdot \nu_n$$

satisfies

$$\lim_{n\to\infty} \|h \cdot \eta_n(h'H) - \eta_n(hh'H)\| = \lim_{n\to\infty} \big\|\sigma(hh')^{-1} h \sigma(h'H) \cdot \nu_n - \nu_n\big\| = 0$$

uniformly on compact sets for $h \in H$ and $h'H \in H/H_i$.

By Proposition 2.1.18, it follows that we can take a sequence of maps such that the convergence holds uniformly on the whole of H/H_i. Hence, identifying $X_i \cong H/H_i$, we find a map $\mu : X_i \to \mathrm{Prob}(H)$ such that

$$\|h \cdot \mu(x) - \mu(hx)\| < \varepsilon$$

for every $h \in K$ and every $x \in X_i$. Now, define $\xi_i : B_i \to \mathrm{Prob}(H)$ by

$$\xi_i(b) = \frac{1}{|\operatorname{supp}_A b|} \sum_{x \in \operatorname{supp}_A b} \mu(x)$$

for $b \in B_i \setminus A^{X_i}$. For $b \in B_i$, set $\xi_i(b) = \mu_0$, where $\mu_0 \in \mathrm{Prob}(H)$ is arbitrary. One easily checks that ξ_i satisfies (3.4.11).

Step 3. $B \wr_X^A H$ **is bi-exact.** Take $\varepsilon > 0$, a compact $C \subseteq B^{X,A}$ and a compact subset $K \subseteq H$. By Lemma 3.1.3 and Corollary 3.4.13, it suffices to prove that there exists a compact $D \subseteq B^{X,A}$ and a Borel map $\zeta : B^{X,A} \to \mathrm{Prob}(H)$ such that

$$\big\|h \cdot \zeta(b) - \zeta\big(\alpha_h(b)\big)\big\| \leqslant \varepsilon \qquad \text{and} \qquad \|\zeta(aba') - \zeta(b)\| \leqslant \varepsilon \tag{3.4.12}$$

for all $h \in K$, $a, a' \in C$ and $b \in B^{X,A} \setminus D$.

By definition of the topology on the semirestricted product $B^{X,A}$, we can take compact sets $C_i \subseteq B_i$ for $i \in I$ such that

$$C \subseteq \prod_{i \in I} C_i$$

and such that $C_i = A^{X_i}$ for all but finitely many $i \in I$. Take $i_1, \ldots, i_n \in I$ such that $C_i = A^{X_i}$ whenever $i \neq i_1, \ldots, i_n$.

For $i = i_1, \ldots, i_n$, the fact that $B \wr_{X_i}^A H$ belongs to class $\mathcal{S}$, allows us to take a compact $D_i \subseteq B_i$ and a Borel map $\zeta_i : B_i \to \mathrm{Prob}(H)$ such that

$$\big\|h \cdot \zeta_i(b) - \zeta_i\big(\alpha_h(b)\big)\big\| \leqslant \varepsilon \qquad \text{and} \qquad \|\zeta_i(aba') - \zeta_i(b)\| \leqslant \varepsilon$$

for $h \in K$, $a, a' \in C_i$ and $b \in B_i \setminus D_i$. By enlarging D_i, we can assume that $A^{X_i} \subseteq D_i$ and $C_i^{-1} A^{X_i} C_i^{-1} \subseteq D_i$. For $i \neq i_1, \ldots, i_n$, we take for $\zeta_i : B_i \to \mathrm{Prob}(H)$ the map ξ_i from step 2 and set $D_i = A^{X_i}$.

For $b \in B^{X,A}$ and $i \in I$, we denote by $b_i \in B^{X_i,A}$ the restriction of b to X_i. We also denote $I_b = \big\{i \in I \bigm| b_i \notin A^{X_i}\big\}$. Define $\zeta : B^{X,A} \to \mathrm{Prob}(H)$ by

$$\zeta_i(b) = \frac{1}{|I_b|} \sum_{i \in I_b} \zeta_i(b_i)$$

for $b \in B^{X,A} \setminus A^X$ and $\zeta_i(b) = \delta_e$ for $b \in A^X$. One easily checks that (3.4.12) holds for $D = \prod_{i \in I} D_i$, since $I_b = I_{aba'}$ for $b \in B^{X,A} \setminus D$ and $a, a' \in C$. $\square$

Chapter 4

Rigidity for von Neumann algebras given by locally compact groups

As highlighted in the introduction, the classification of group von Neumann algebras and group measure space von Neumann algebras is a difficult problem. In the last two decades, due to Popa's discovery of deformation/rigidity theory [Pop06a; Pop06b; Pop06c] a lot of progress has been made in the case of countable groups.

By Singer's theorem (see Theorem 2.5.23), the classification problem for group measure space factors $M = L^\infty(X) \rtimes \Gamma$ for free ergodic probability measure preserving (pmp) actions of a countable groups $\Gamma \curvearrowright (X, \mu)$ splits into two separate problems: the uniqueness problem for the Cartan subalgebra $L^\infty(X)$ and the classification problem for $\Gamma \curvearrowright (X, \mu)$ up to orbit equivalence. In recent years, striking progress has been made on both problems. In [OP10b], it is proved that for *profinite* free ergodic pmp actions of the free groups $\mathbb{F}_n$, the crossed product M has a unique Cartan subalgebra up to unitary conjugacy. In [CS13], it was shown that the same holds for profinite actions of nonamenable, weakly amenable groups in class $\mathcal{S}$ (thus in particular for nonelementary hyperbolic groups). For *arbitrary* free ergodic pmp actions of the same groups, the uniqueness of the Cartan subalgebra was established in [PV14a; PV14b].

In this chapter, we prove such results for *locally compact* groups with these properties. In Section 4.3 we prove that for any nonamenable, weakly amenable, locally compact group G in class $\mathcal{S}$ and for any free, pmp action $G \curvearrowright (X,\mu)$, the group measure space von Neumann algebra $M = L^\infty(X) \rtimes G$ has a unique Cartan subalgebra up to unitary conjugacy (Theorem F from the introduction). By Proposition C proved in the previous chapter, along with Theorem 2.1.8 and Propositions 2.1.7, 2.2.4 and 3.4.1, this class of groups includes all real rank one simple Lie groups with finite center, as well as all locally compact groups acting continuously and properly on a tree or a hyperbolic graph.

As explained in Section 2.5.5, it is important to note that for nondiscrete groups G the Cartan subalgebra of $M = L^\infty(X) \rtimes G$ is not $L^\infty(X)$, since there is no faithful, normal conditional expectation $M \to L^\infty(X)$. Instead, the canonical Cartan subalgebra can be found by choosing a cross section for $G \curvearrowright (X,\mu)$ as in Proposition 2.5.41.

This Cartan rigidity result is deduced from Theorem 4.2.2, a very general structural result, similar to [PV14b, Theorem 3.1], on the normalizer $\mathcal{N}_{pMp}(A)$ of a von Neumann subalgebra $A \subseteq pMp$, when M is equipped with a coaction $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ of a locally compact, weakly amenable group in class $\mathcal{S}$. By adapting the methods of [BHV18] to this abstract setting, we were even able to obtain a similar result on the stable normalizer $\mathcal{N}^{\mathrm{s}}_M(A)$ when the group G has CMAP (Theorem 4.5.1).

In Section 4.4, we turn to orbit equivalence rigidity. We first prove a cocycle superrigidity result for arbitrary cocycles of irreducible pmp actions $G_1 \times G_2 \curvearrowright (X,\mu)$ taking values in a locally compact group with property (S) (see Theorem 4.4.1). This result is very similar to the cocycle superrigidity theorem of [MS04], where the target group is assumed to be a closed subgroup of the isometry group of a negatively curved space. We then deduce that Sako's orbit equivalence rigidity theorem [Sak09b] for irreducible pmp actions $G_1 \times G_2 \curvearrowright (X,\mu)$ of nonamenable groups in class $\mathcal{S}$ stays valid in the locally compact setting (see Theorem 4.4.2). Recall here that a nonsingular action $G_1 \times G_2 \curvearrowright (X,\mu)$ of a direct product group is called irreducible if both G_1 and G_2 act ergodically.

Combining the Cartan rigidity result with the orbit equivalence result above, we deduce the W*-strong rigidity result Theorem G: if $G = G_1 \times G_2$ and $H = H_1 \times H_2$ are products of nonamenable, unimodular locally compact groups without nontrivial compact normal subgroups with H_1, H_2 weakly amenable and in class $\mathcal{S}$, and if moreover $G \curvearrowright (X,\mu)$ and $H \curvearrowright (Y,\eta)$ are free, pmp and irreducible actions, then an isomorphism $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$ implies conjugacy of the actions.

Ozawa proved in [Oza04] that the group von Neumann algebra $L(\Gamma)$ is solid for Γ a countable discrete group in class $\mathcal{S}$, i.e. for every diffuse von

Neumann subalgebra $A \subseteq L(\Gamma)$ (which is automatically with expectation), the relative commutant $A' \cap L(\Gamma)$ is amenable. It is not difficult to prove that this result remains true for locally compact groups in class $\mathcal{S}$ (see Proposition 4.6.1). Using the results on the normalizer and the stable normalizer mentioned above (see Theorems 4.2.2 and 4.5.1), we prove the *strong solidity* result Theorem H from the introduction in Section 4.6. Recall that a von Neumann algebra M is strongly solid if for every diffuse, amenable von Neumann subalgebra $A \subseteq M$ with expectation, the normalizer $\mathcal{N}_A(M)'' = \{u \in \mathcal{U}(M) \mid uAu^* = A\}''$ remains amenable. A von Neumann algebra M is stably strongly solid if the same happens for the stable normalizer $\mathcal{N}^{\mathrm{s}}_M(A)'' = \{x \in M \mid xAx^* \subseteq A \text{ and } x^*Ax \subseteq A\}''$ of diffuse, amenable von Neumann subalgebras A with expectation. See Section 2.4.9 for more details. We prove that all finite trace corners of diffuse group von Neumann algebras of unimodular, weakly amenable groups in class $\mathcal{S}$ are strongly solid. For locally compact groups in class $\mathcal{S}$ with the complete metric approximation property (CMAP) for which the group von Neumann algebra is diffuse, we obtain that the group von Neumann algebra is stably strongly solid.

The solidity result Proposition 4.6.1 above implies in particular that if G is a locally compact group in class $\mathcal{S}$ such that $L(G)$ is a nonamenable factor, then $L(G)$ is prime, i.e. $L(G)$ can not be decomposed as a tensor product $M_1 \otimes M_2$ for factors M_1 and M_2 that are not of type I. Combining our characterization of class $\mathcal{S}$ in terms of an amenable action on a compactification that is small at infinity (see Theorem B) with the unique prime factorization results of Houdayer and Isono in [HI17] along with the generalization [AHHM18, Application 4] by Ando, Haagerup, Houdayer, and Marrakchi, we prove the unique prime factorization result Theorem I for (tensor products of) such group von Neumann algebras in Section 4.7: if $G = G_1 \times \cdots \times G_n$ is a direct product of locally compact groups in class $\mathcal{S}$ whose group von Neumann algebras are nonamenable factors and if $N = N_1 \otimes \cdots \otimes N_m$ is a tensor product of factors that are not of type I, then an isomorphism

$$L(G) = L(G_1) \otimes \cdots \otimes L(G_n) \cong N_1 \otimes \cdots \otimes N_m = N$$

implies that $m \leqslant n$. If moreover the factors N_i are prime, then $m = n$ and (after relabeling) $L(G_i)$ is stably isomorphic to N_i for $i = 1, \dots, n$. This result is a locally compact version of the unique prime factorization result by Ozawa and Popa in [OP04].

It is important to note that for many locally compact groups G in class $\mathcal{S}$, the group von Neumann algebra $L(G)$ is amenable or even type I. For instance, the group von Neumann algebra of a connected group is always amenable by [Con76, Corollary 6.9]. See also Section 2.4.8 for a discussion of this phenomenon. However, the following group G due to Suzuki provides an example of a locally compact group whose group von Neumann algebra $L(G)$ is a nonamenable type II_∞ factor.

Example (Suzuki) Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{F}_2$ by flipping the generators. Then the compact group $K = \prod_{k\in\mathbb{N}} \mathbb{Z}_2$ acts on the infinite free product $H = *_{k\in\mathbb{N}} \mathbb{F}_2$ by letting the k^{th} component of K flip the generators in the k^{th} component of H. The semidirect product $G = H \rtimes K$ satisfies the conditions of [Suz16, Proposition] with $K_n = \prod_{k=n+1}^{\infty} \mathbb{Z}_2$ and $L_n = (*_{k=0}^{n} \mathbb{F}_2) \rtimes K$. Hence, by [Suz16, section on group von Neumann algebras], its group von Neumann algebra is a nonamenable factor of type II_∞. Moreover, G has CMAP and belongs to class $\mathcal{S}$ since the cocompact subgroup H does.

Furthermore, certain classes of groups acting on trees have nonamenable group von Neumann algebras by [HR19, Theorem C and D]. Also, [Rau19b, Theorem E and F] would provide conditions on such a group G under which $L(G)$ would be a nonamenable factor. In particular, for every $q \in \mathbb{Q}$ with $0 < q < 1$ [Rau19b, Theorem G] would provide examples of groups in class $\mathcal{S}$ for which the group von Neumann algebra would be a nonamenable factor of type III_q. However, due to a mistake in [Rau19b, Lemma 5.1], there is a gap in the proofs of these results (see also [Rau19a, p 20]), and it is currently not completely clear whether these results hold as stated there.

Most of this chapter is based on the author's joint publication [BDV18] with Arnaud Brothier and Stefaan Vaes. Section 4.7 is based on the author's publication [Dep19].

4.1 Preliminaries

We start this chapter by introducing several concepts of von Neumann algebras that we will need to state or proof the results in this chapter.

4.1.1 Coactions

As explained in the introduction, we will deduce the main results of this chapter from a general result on the (stable) normalizer inside a semifinite von Neumann algebra equipped with a so-called coaction. The concept of a coaction comes from the theory of locally compact quantum groups. In this section, we recall its definition and discuss the properties that we will need in this rest of this chapter. For a more detailed discussion and motivation of the theory of locally compact quantum groups and their (co)actions, we refer to [BS93; Vae01; BSV03; Van05; KV03].

Let G be a locally compact group. Its group von Neumann algebra $L(G)$ carries a normal, unital $*$-morphism $\Delta : L(G) \to L(G) \mathbin{\overline{\otimes}} L(G)$ satisfying $\Delta(u_g) = u_g \otimes u_g$ for all $g \in G$. Indeed, consider the unitary $U \in L^\infty(G) \mathbin{\overline{\otimes}} L(G)$ given by

$$(U\xi)(s,t) = \xi(s, s^{-1}t). \tag{4.1.1}$$

for all $\xi \in L^2(G) \otimes L^2(G)$ and a.e. $s, t \in G$. Then, Δ is defined by

$$\Delta(x) = U(x \otimes 1)U^*$$

for $x \in L(G)$. This map is called the *comultiplication* on $L(G)$. Since

$$(\Delta \otimes \mathrm{id})\big(\Delta(u_g)\big) = u_g \otimes u_g \otimes u_g = (\mathrm{id} \otimes \Delta)\big(\Delta(u_g)\big)$$

for $g \in G$, the map Δ satisfies the so-called coassociativity relation $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

A coaction is now defined as follows.

Definition 4.1.1. Let G be a locally compact group and M a von Neumann algebra. A (right) coaction of G on M is an injective, unital, normal $*$-morphism $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ satisfying $(\Phi \otimes \mathrm{id})\Phi = (\mathrm{id} \otimes \Delta)\Phi$.

Note that the comultiplication Δ is a coaction of G on $L(G)$. We will now give two other examples of coactions that we will encounter in this text.

Example 4.1.2. Let $M = L^\infty(X) \rtimes G$ be the group measure space von Neumann algebra associated to a nonsingular action $G \curvearrowright (X, \mu)$. Consider the canonical representation of M on $\mathcal{K} = L^2(G) \otimes L^2(X)$ and let U be the unitary from (4.1.1). We define $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ by

$$\Phi(x) = U_{13}(x \otimes 1)U_{13}^*,$$

where we use the leg numbering notation for tensor products to denote the unitary $U_{13} : L^2(G) \otimes L^2(X) \otimes L^2(G) \to L^2(G) \otimes L^2(X) \otimes L^2(G)$ acting in the first and the third leg.

A straightforward calculation yields that $\Phi(u_g f) = u_g f \otimes u_g \in M \mathbin{\overline{\otimes}} L(G)$ for all $g \in G$ and $f \in L^\infty(X)$ and hence that Φ is indeed a well defined coaction.

Example 4.1.3. Let $\mathcal{R}$ be a countable equivalence relation on (X, μ) and $\omega : \mathcal{R} \to G$ a cocycle into a locally compact group. We associate to $M = L(\mathcal{R})$ and ω the following coaction $\Phi_\omega : M \to M \mathbin{\overline{\otimes}} L(G)$.

Let $U_\omega : L^2(\mathcal{R}) \otimes L^2(G) \to L^2(\mathcal{R}) \otimes L^2(G)$ be the unitary defined by

$$(U_\omega \xi)(x, y, g) = \xi\big(x, y, \omega(y, x)g\big)$$

for a.e. $(x,y) \in \mathcal{R}$ and $g \in G$. Define Φ_ω by

$$\Phi_\omega(x) = U_\omega(x \otimes 1)U_\omega^*$$

for $x \in M$. A straightforward calculation shows that for $f \in L^\infty(X)$ and $\varphi \in [\mathcal{R}]$, we have

$$\big(\Phi_\omega(f)\xi\big)(x,y,g) = f(x)\xi(x,y,g) \tag{4.1.2}$$

and

$$\big(\Phi_\omega(u_\varphi)\xi\big)(x,y,g) = \xi\big(\varphi^{-1}(x), y, \omega(\varphi^{-1}(x),x)g\big) \tag{4.1.3}$$

for all $\xi \in L^2(\mathcal{R}) \otimes L^2(G)$ and a.e. $x, y \in X$ and a.e. $g \in G$. Hence,

$$\Phi_\omega(f) = f \otimes 1 \qquad \text{and} \qquad \Phi_\omega(u_\varphi) = (u_\varphi \otimes 1)V_\varphi, \tag{4.1.4}$$

for $f \in L^\infty(X)$ and $\varphi \in [\mathcal{R}]$ where $V_\varphi \in L^\infty(X) \mathbin{\overline{\otimes}} L(G)$ is defined by $V_\varphi(x) = \lambda_{\omega(\varphi(x),x)}$ for $x \in X$. In particular, $\Phi_\omega(x) \in M \mathbin{\overline{\otimes}} L(G)$ for all $x \in M$, since M is generated by $L^\infty(X)$ and u_φ for $\varphi \in [\mathcal{R}]$.

To see that Φ_ω is a coaction, note that

$$(\Phi_\omega \otimes 1)\big(\Phi_\omega(f)\big) = f \otimes 1 \otimes 1 = (\mathrm{id} \otimes \Delta)\big(\Phi_\omega(f)\big)$$

for $f \in L^\infty(X)$. On the other hand, fix $\varphi \in [\mathcal{R}]$ and define $V = V_\varphi$ as above. Since $V \in L^\infty(X) \mathbin{\overline{\otimes}} L(G)$, we have $(\Phi_\omega \otimes 1)(V) = V_{13}$, where we used the leg numbering notation. Hence,

$$(\Phi_\omega \otimes 1)\big(\Phi_\omega(u_\varphi)\big) = (u_\varphi \otimes 1 \otimes 1)V_{12}V_{13} = (u_\varphi \otimes 1 \otimes 1)W = (\mathrm{id} \otimes \Delta)\big(u_\varphi\big),$$

where $W \in L^\infty(X) \mathbin{\overline{\otimes}} L(G) \mathbin{\overline{\otimes}} L(G)$ is given by

$$W(x) = \lambda_{\omega(\varphi(x),x)} \otimes \lambda_{\omega(\varphi(x),x)} = \Delta(\lambda_{\omega(\varphi(x),x)})$$

for a.e. $x \in X$. Again, since M is generated by $L^\infty(X)$ and u_φ for $\varphi \in [\mathcal{R}]$, it follows that $(\Phi_\omega \otimes \mathrm{id}) \circ \Phi_\omega = (\mathrm{id} \otimes \Delta) \circ \Phi_\omega$.

Every coaction has a unitary implementation in the following sense. This implementation was first obtained by Vaes in [Vae01]. An earlier attempt at a proof can be found in [Sau85]. However, that proof contains a mistake. Let M be a von Neumann algebra with a coaction $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ of a locally compact group G. Suppose that M is in standard representation on $\mathcal{H}$ and let $J : \mathcal{H} \to \mathcal{H}$ be the associated modular conjugation. Denote by $\hat{J} : L^2(G) \to L^2(G)$ the anti-unitary operator given by $\hat{J}\xi = \overline{\xi}$ for $\xi \in L^2(G)$. The following is [Vae01, Proposition 3.7, Proposition 3.12 and Theorem 4.4].

Theorem 4.1.4. *Let G be a locally compact group, M a von Neumann algebra and $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ a coaction. Then, there exists a unitary $V \in B(\mathcal{H}) \mathbin{\overline{\otimes}} L(G)$ satisfying*

(a) $\Phi(x) = V(x \otimes 1)V^*$ *for every* $x \in M$,

(b) $V(J \otimes \hat{J}) = (J \otimes \hat{J})V^*$.

Moreover, V is a corepresentation in the sense that $(\mathrm{id} \otimes \Delta)(V) = V_{12}V_{13}$.

By [dCES79, Théorème 2.10] and [Eym64, Théorème 3.34] (see also [Sau85, Remarque 2.2]), the fact that V is a corepresentation yields a nondegenerate $*$-morphism $\pi : C_0(G) \to B(\mathcal{H})$ given by

$$\pi\big((\mathrm{id} \otimes \omega)(U)\big) = (\mathrm{id} \otimes \omega)V, \tag{4.1.5}$$

where U is again the unitary from (4.1.1).

4.1.2 The module $\mathcal{H} \mathbin{\overline{\otimes}} M$

Let (M, τ) be a tracial von Neumann algebra and $\mathcal{H}$ a Hilbert space. Denote by $\hat{x}$ the image of x under the canonical injection $M \hookrightarrow L^2(M)$. The Hilbert space $\mathcal{K} = \mathcal{H} \otimes L^2(M)$ is a right M-module for the operation defined by $(\xi \otimes \hat{x})y = \xi \otimes \widehat{xy}$. We denote the module of right bounded vectors $\mathcal{K}^0$ by $\mathcal{H} \mathbin{\overline{\otimes}} M$.

As in Section 2.4.7, the module $\mathcal{H} \mathbin{\overline{\otimes}} M$ carries an M-valued inner product given by

$$\langle \xi \otimes \hat{x}, \eta \otimes \hat{y} \rangle_M = L^*_{\xi \otimes \hat{x}} L_{\eta \otimes \hat{y}} = \langle \xi, \eta \rangle\, y^* x$$

We identify a vector $\xi \in \mathcal{H}$ with the linear operator

$$\mathbb{C} \to \mathcal{H} : \lambda \mapsto \lambda\xi$$

and we denote by $\bar{\eta}$ for $\eta \in \mathcal{H}$ the operator

$$\mathcal{H} \to \mathbb{C} : \zeta \mapsto \langle \zeta, \eta \rangle .$$

Denote by $\mathcal{M}_0$ the linear span of the operators

$$\begin{pmatrix} T \otimes x_{11} & \xi \otimes x_{12} \\ \overline{\eta} \otimes x_{21} & x_{22} \end{pmatrix} \in B\big(\mathcal{K} \oplus L^2(M)\big)$$

for $x_{11}, x_{12}, x_{21}, x_{22} \in M$, $T \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$. Let $\mathcal{M}$ be the w.o. closure of $\mathcal{M}_0$. Identifying $\mathcal{H} \mathbin{\overline{\otimes}} M$ with the space $B\big(L^2(M)_M, \mathcal{K}_M\big)$ of bounded operators $L^2(M) \to \mathcal{K}$ that commute with the right M-module operation, we have $\mathcal{H} \mathbin{\overline{\otimes}} M = e\mathcal{M}f$, where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We will use the following version of the Kaplansky density theorem.

Theorem 4.1.5 (Kaplansky density theorem)**.** *Let M be a von Neumann algebra and $\mathcal{H}$ a Hilbert space. Suppose that $M_0 \subseteq M$ is a w.o. dense $*$-subalgebra and $\mathcal{H}_0$ a dense subspace. Then, the $\|.\|_\infty$-unit ball of $\mathcal{H}_0 \otimes_{alg} M_0$ is strong* operator dense in the $\|.\|_\infty$-unit ball of $\mathcal{H} \mathbin{\overline{\otimes}} M$.*

Proof. For $\xi, \eta \in \mathcal{H}_0$, we denote $T_{\xi,\eta} \in B(\mathcal{H})$ the finite rank operator

$$T_{\xi,\eta} : \mathcal{H} \to \mathcal{H} : \zeta \mapsto \langle \zeta, \eta \rangle \, \xi.$$

Set $N_0 = \operatorname{span}\{T_{\xi,\eta}\}_{\xi,\eta \in \mathcal{H}_0}$. Since $\mathcal{H}_0$ is dense in $\mathcal{H}$, we have that N_0 is w.o. dense in $B(\mathcal{H})$. Define the $*$-subalgebra $\mathcal{M}_1 \subseteq \mathcal{M}_0$ as the linear span of the operators

$$\begin{pmatrix} T \otimes x_{11} & \xi \otimes x_{12} \\ \overline{\eta} \otimes x_{21} & x_{22} \end{pmatrix} \in B\big(\mathcal{K} \oplus L^2(M)\big)$$

for $x_{11}, x_{12}, x_{21}, x_{22} \in M_0$, $T \in N_0$ and $\xi, \eta \in \mathcal{H}_0$. Then, $\mathcal{M}_1$ is a w.o. dense $*$-subalgebra of $\mathcal{M}$. Hence, by the (classical) Kaplansky density theorem (see Theorem 2.4.3), we have that the unit ball of $\mathcal{M}_1$ is strong* operator dense in the unit ball $\mathcal{M}$. It follows that the unit ball of $\mathcal{H}_0 \otimes_{\text{alg}} M_0 = e\mathcal{M}_1 f$ is strong* operator dense in the unit ball of $\mathcal{H} \mathbin{\overline{\otimes}} M = e\mathcal{M}f$. ☐

4.1.3 Second dual

Let M be any von Neumann algebra. Consider the following representation

$$\pi = \bigoplus_{\varphi \in M_+^*} \pi_\varphi, \tag{4.1.6}$$

where M_+^* denotes the set of (not necessarily normal) positive linear functionals and π_φ denotes the GNS-representation of M with respect to φ. Note that since the φ are not required to be normal, also the representations π_φ are not. Hence, $\pi(M)$ is not a von Neumann algebra. We can however consider the von Neumann algebra $\widetilde{M} = \pi(M)''$ generated by the image of this representation. This von Neumann algebra satisfies the following universal property. A proof can be found in [Tak79, Theorem III.2.4].

Theorem 4.1.6. *Let M be a von Neumann algebra. Let π be the representation defined in (4.1.6) and denote $\widetilde{M} = \pi(M)''$. For every representation $\rho : M \to B(\mathcal{H})$ there exists a unique normal unital $*$-morphism $\tilde{\rho} : \widetilde{M} \to \rho(M)''$ such that $\rho = \tilde{\rho} \circ \pi$.*

$$\begin{array}{ccc} M & \xrightarrow{\ \rho\ } & B(\mathcal{H}) \\ {\scriptstyle \pi}\big\downarrow & \overset{\exists! \tilde{\rho}}{\nearrow} & \\ \tilde{M} & & \end{array}$$

This yields the following definition.

Definition 4.1.7. Let M be a von Neumann algebra. The representation π defined in (4.1.6) is called the *universal representation* of M and the von Neumann algebra $\widetilde{M} = \pi(M)''$ is called the *universal enveloping von Neumann algebra* of M.

Since $\pi(M) \subseteq \pi(M)''$ is a $*$-subalgebra, every normal linear functional on $\widetilde{M}$ restricts to a bounded linear functional on M. This yields a map $\rho : \widetilde{M}_* \to M^*$. This map is actually an actually an isometric isomorphism (see for instance [Bla06, Proposition III.5.2.6]). Dualizing then gives the following result (see also [Tak79, Theorem III.2.4]). This result was first obtained by Sherman and Takeda [She50; Tak54].

Theorem 4.1.8. *Let M be a von Neumann algebra. The universal enveloping von Neumann algebra $\widetilde{M}$ is isomorphic (as a Banach space) to the second dual M^{**}.*

In particular, the second dual M^{**} can be viewed as a von Neumann algebra. The map ρ identifies the (not necessarily normal) linear functionals on M with the normal linear functionals on M^{**}. As with any Banach space, we have that the unit ball $(M)_1$ is weak* dense in the unit ball $(M^{**})_1$. In the language of topologies on von Neumann algebras, this yields that M is a w.o. dense $*$-subalgebra of M^{**}.

Dualizing the inclusion $M_* \hookrightarrow M^*$ yields a normal $*$-morphism $M^{**} \to M$ that is the identity on M. Denote by $z \in \mathcal{Z}(M^{**})$ the support of this morphism. We have the following result. A proof can be found in [Tak03a, Definition III.2.15].

Theorem 4.1.9. *Let M be a von Neumann algebra. Let $z \in \mathcal{Z}(M^{**})$ be the support of the natural $*$-morphism $M^{**} \to M$. Then, for any $\varphi \in M^* \cong (M^{**})_*$, we have that the linear functional*

$$\tilde{\varphi} : M \to \mathbb{C} : x \mapsto \varphi(xz)$$

is normal. Moreover, if $\varphi \in M_*$, then $\tilde{\varphi} = \varphi$.

The following theorem is easily verified and can also be found in [Bla06, p. III.5.2.10].

Theorem 4.1.10. *Let M and N be two von Neumann algebras. If $\Phi : M \to N$ is a bounded linear map, then the second dual map $\Phi^{**} : M^{**} \to N^{**}$ is a* normal *linear map with the same norm satisfying $\Phi^{**}|_M = \Phi$. Moreover, Φ^{**} has one of the properties injectivity, surjectivity, (complete) positivity, being a $*$-morphism and being a conditional expectation if and only if Φ has the same property.*

Using the identification $M^* \cong (M^{**})_*$, we can also define the polar decomposition of a non-normal linear functional φ on M.

Theorem 4.1.11. *Let M be a von Neumann algebra and $\varphi \in M^*$. Then, there exists a unique positive linear functional $\psi \in M^*$ such that $\psi(x) = \varphi(xu)$ and $\varphi(x) = \psi(xu^*)$ for some partial isometry $u \in M^{**}$. Moreover, $\|\psi\| = \|\varphi\|$.*

As in Theorem 2.4.6, we denote this unique positive linear functional by $|\varphi|$. By Theorems 2.4.7 and 4.1.10, we have the following.

Theorem 4.1.12. *Let $\varphi : M \to \mathbb{C}$ be a normal linear functional and $\beta \in$* $\mathrm{Aut}(M)$*. Then,*

$$|\varphi \circ \beta| = |\varphi| \circ \beta.$$

4.2 A dichotomy for von Neumann algebras with a coaction

In this section, we prove a very general dichotomy result for the normalizer $\mathcal{N}_{pMp}(A)$ of a von Neumann subalgebra $A \subseteq pMp$ when M is a semifinite von Neumann algebra equipped with a coaction $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ of a locally compact, weakly amenable group G in class $\mathcal{S}$. We define the following notions.

Definition 4.2.1. Let (M, Tr) be a semifinite von Neumann algebra. Let $p \in M$ be a projection with $\mathrm{Tr}(p) < +\infty$ and $A \subseteq pMp$ a von Neumann subalgebra. Consider the pMp-M-bimodule $\mathcal{H} = \Phi(p)\big(L^2(M) \otimes L^2(G)\big)$ given by $x \cdot \xi \cdot y = \Phi(x)\xi(y \otimes 1)$. We say that

(a) A can be Φ-*embedded* if the pMp-M-bimodule $\mathcal{H}$ admits a nonzero A-central vector,

(b) A is Φ-amenable if $\mathcal{H}$ is left A-amenable, i.e. if there exists a nonzero positive linear functional Ω on $\Phi(p)\big(M \mathbin{\overline{\otimes}} B(L^2(G))\big)\Phi(p)$ that is $\Phi(A)$-central and satisfies $\Omega(\Phi(x)) = \mathrm{Tr}(x)$ for all $x \in pMp$.

We are now ready to state our dichotomy type theorem. It is a more abstract version of [PV14b, Theorem 3.1].

Theorem 4.2.2. *Let G be a weakly amenable, locally compact group in class $\mathcal{S}$. Let (M, Tr) be a von Neumann algebra with a faithful normal semifinite trace and $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ a coaction. Let $p \in M$ be a projection with $\mathrm{Tr}(p) < \infty$ and $A \subseteq pMp$ a von Neumann subalgebra.*

If A is Φ-amenable then at least one of the following statements holds: A can be Φ-embedded or $\mathcal{N}_{pMp}(A)''$ stays Φ-amenable.

Outline of the proof The proof follows closely the proofs of [PV14a, Theorem 5.1] and [PV14b, Theorem 3.1]. The main novelty is to develop, in the context of coactions of locally compact groups, a framework in which the main ideas of [PV14a; PV14b] are applicable.

For the reader's convenience, we first outline the big steps in the proof of Theorem 4.2.2. Let G, M, Φ and A be as in the statement of the theorem. First, we deduce from the weak amenability of the group G the existence of a net $(\eta_i)_i$ in $A_c(G)$, the compactly supported functions in the Fourier algebra $A(G)$, such that the associated maps $m_i = (\mathrm{id} \otimes \eta_i) \circ \Delta : L(G) \to L(G)$ and $\varphi_i = (\mathrm{id} \otimes \eta_i) \circ \Phi : M \to M$ satisfy $\varphi_i \to \mathrm{id}$ in pointwise s.o. topology and $\mathrm{id}_{\mathcal{H}} \otimes m_n \to \mathrm{id}$ in pointwise s.o. topology for every Hilbert space $\mathcal{H}$ (Lemma 4.2.5). Similar to [PV14a, Lemma 5.2], we then deduce in Lemma 4.2.7 from the Φ-amenability of $A \subseteq pMp$ that the net of maps

$$\psi_i : pMp \to pMp : x \mapsto p\varphi(x)p$$

is *adapted* with respect to A in the sense of Definition 4.2.6.

Consider the M-pMp-bimodule $\mathcal{K} = L^2(Mp) \otimes L^2(G)$ given by $x \cdot \xi \cdot y = \Phi(x)\xi(y \otimes 1)$ and denote by $\rho : pMp \to B(\mathcal{K})$ the normal $*$-antimorphism given by the right action. We define $\mathcal{N}$ as the von Neumann algebra generated by $\Phi(M)$ and $\rho(A)$. Let $q = \Phi(p)$. Similar to [PV14a, Proposition 5.4] and [OP10b, Proposition 7], we construct a bounded net of normal linear functionals $\mu_i : q\mathcal{N}q \to \mathbb{C}$ that approximates the trace on M and on $\rho(A)$, that is approximately invariant under conjugating both $\rho(A)$ and $\Phi(M)$ by a unitary $u \in \mathcal{N}_{pMp}(A)$ and that is approximately invariant under translating μ_i by $\Phi(u^*)\rho(u)$ for $u \in \mathcal{U}(A)$. (see step 1 on p. 154).

From this we deduce an analog of [PV14a, Theorem 5.1], i.e. we deduce the existence of a net of *positive* normal linear functionals $\omega_i : q\mathcal{N}q \to \mathbb{C}$ that still satisfies similar properties (see step 2 on p. 156). By representing $\mathcal{N}$ on a standard Hilbert space $\mathcal{H}$, we then deduce the existence of a net of approximately tracial vectors $(\xi_i)_i$ with similar invariance properties (see step 3 on p. 158).

Using the canonical unitary implementation of the coaction $\Psi : \mathcal{N} \to \mathcal{N} \mathbin{\overline{\otimes}} L(G)$ given by $\Psi = \mathrm{id} \otimes \Delta$, we get a nondegenerate $*$-morphism $\pi : C_0(G) \to B(\mathcal{H})$ that allows the formulation of the dichotomy: either $\limsup_i \|\pi(F)\xi_i\| = 0$ for all $F \in C_0(G)$, or there exists an $F \in C_0(G)$ such that $\limsup_i \|\pi(F)\xi_i\| > 0$. In the first case, we deduce from the fact that G is in class $\mathcal{S}$ that $\mathcal{N}_{pMp}(A)''$ stays Φ-amenable. In the second case, we deduce that A can be Φ-embedded.

Proof of Theorem 4.2.2 For the rest of this section, fix a weakly amenable, locally compact group G in class $\mathcal{S}$ and a semifinite von Neumann algebra (M, Tr) with coaction $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$. We denote by $\Lambda_G < +\infty$ the Cowling-Haagerup constant of G. Let $\Delta : L(G) \to L(G) \mathbin{\overline{\otimes}} L(G)$ be the comultiplication as in (4.1.1).

Denote by $A(G)$ the Fourier algebra of G and by $A_c(G) \subseteq A(G)$ the subalgebra of compactly supported functions in $A(G)$. Identifying $A(G)$ with the predual $L(G)_*$, the associated map $m_\varphi : L(G) \to L(G)$ as in Proposition 2.4.32 can be expressed using the comultiplication.

Lemma 4.2.3. *Let $\eta \in A(G)$. The map $m_\eta = (\mathrm{id} \otimes \eta) \circ \Delta : L(G) \to L(G)$ is normal and satisfies $m_\eta(u_g) = \eta(g)u_g$ for all $g \in G$.*

Proof. We have

$$m_\eta(u_g) = (\mathrm{id} \otimes \eta)(u_g \otimes u_g) = \eta(g)u_g.$$

□

Using the coaction, we can associate to an $\eta \in A(G)$ the normal completely bounded map

$$\varphi_\eta : M \to M : x \mapsto (\mathrm{id} \otimes \eta)\big(\Phi(x)\big). \tag{4.2.1}$$

The following lemma proves the relationship with the associated completely bounded map $m_\eta : L(G) \to L(G)$

Lemma 4.2.4. *Let $\eta \in A(G)$. Denote by $m_\eta : L(G) \to L(G)$ the completely bounded map as in Lemma 4.2.3 and by $\varphi_\eta : M \to M$ the completely bounded map as in (4.2.1). Then,*

$$\Phi \circ \varphi_\eta = (\mathrm{id} \otimes m_\eta) \circ \Phi$$

Moreover, $\|\varphi_\eta\|_{cb} \leqslant \|m_\eta\|_{cb}$.

Proof. We have

$$\Phi \circ \varphi_\eta = \Phi \circ (\mathrm{id} \otimes \eta) \circ \Phi = (\mathrm{id} \otimes \mathrm{id} \otimes \eta) \circ (\Phi \otimes \mathrm{id}) \circ \Phi$$

$$= (\mathrm{id} \otimes \mathrm{id} \otimes \eta) \circ (\mathrm{id} \otimes \Delta) \circ \Phi = (\mathrm{id} \otimes m_\eta) \circ \Phi.$$

Since Φ is faithful, Φ and all its amplifications $\Phi^{(n)} = \Phi \otimes \mathrm{id}_{M_n(\mathbb{C})}$ are isometric, and thus $\|\varphi_\eta\|_{\mathrm{cb}} \leqslant \|m_\eta\|_{\mathrm{cb}}$. □

We also have the following lemma.

Lemma 4.2.5. *There exists a net $(\eta_i)_i$ in $A_c(G)$ with $\|\eta_i\|_{cb} \leqslant \Lambda_G$ for all i and such that for the associated completely bounded maps $m_i : L(G) \to L(G)$, we have*

$$(\mathrm{id} \otimes m_i)(T) \to T$$

in s.o. topology for all $T \in B(\mathcal{H}) \mathbin{\overline{\otimes}} L(G)$. Moreover, for the associated completely bounded maps $\varphi_i : M \to M$ as in (4.2.1), we have

$$\varphi_i(x) \to x$$

in s.o. topology for all $x \in M$.

Proof. By Proposition 2.1.6, we can take a sequence $(\tilde{\eta}_n)_n$ in $A_c(G)$ with $\|\tilde{\eta}_n\|_{\mathrm{cb}} < \Lambda_G$ for all $n \in \mathbb{N}$ and such that $\eta_n \to 1$ uniformly on compact sets. Since $A_c(G)$ is dense in $A(G)$, it follows that $\|\eta_n u - u\|_A \to 0$ for every $u \in A(G)$. Denote by $\widetilde{m}_n : L(G) \to L(G)$ the associated completely bounded maps.

Without loss of generality, we can assume that $\mathcal{H} = \ell^2(\mathbb{N})$. We claim that $(\mathrm{id} \otimes \widetilde{m}_n)(T) \to T$ in the w.o. topology topology for every $T \in B(\mathcal{H}) \mathbin{\overline{\otimes}} L(G)$. Since $\|(\mathrm{id} \otimes \widetilde{m}_n)(T)\| \leqslant \Lambda_G \|T\| < +\infty$, it suffices to prove that every matrix coefficient $(\mathrm{id} \otimes \widetilde{m}_n)(T)_{i,j} = \widetilde{m}_n(T_{i,j}) \to T_{i,j}$ in w.o. topology. But, identifying $A(G) = L(G)_*$ and using that m_n is the dual map of the map $A(G) \to A(G) : u \mapsto \eta_n u$, we have $\varphi(m_n(x) - x) \to 0$ for every $x \in L(G)$ and hence $m_n(x) \to x$ in w.o. topology.

Now, $\mathrm{id} \otimes \mathrm{id}$ is in the pointwise-w.o. closure of all $\{\mathrm{id} \otimes \widetilde{m}_n\}_{n \geqslant n_0}$ for $n_0 \in \mathbb{N}$. Hence, for all $n_0 \in \mathbb{N}$ it is in the pointwise-s.o. closure of the convex hull of $\{\mathrm{id} \otimes \widetilde{m}_n\}_{n \geqslant n_0}$. Hence, we can take a net $(\eta_i)_i$ of convex combinations of $\{\widetilde{m}_n\}_{n \in \mathbb{N}}$ such that for the net of associated normal completely bounded maps $(m_i)_i$ we have

$$(\mathrm{id} \otimes m_i)(T) \to T$$

in s.o. topology for all $T \in B(\mathcal{H}) \mathbin{\overline{\otimes}} L(G)$.

Finally, to prove the convergence of $(\varphi_i)_i$, note that since Φ is normal, $\Phi(M)$ is a von Neumann algebra and hence $\Phi : M \to \Phi(M)$ is a *-isomorphism between von Neumann algebras. So, Φ is a homeomorphism for the s.o. topology on norm bounded sets, and hence Lemma 4.2.4 implies that $\varphi_i(x) \to x$ in s.o. topology for every $x \in M$. □

Whenever $\mathcal{V}$ is a set of operators on a Hilbert space, we denote by $[\mathcal{V}]$ the operator norm closed linear span of $\mathcal{V}$. We define the following.

Definition 4.2.6. Let M, G and Φ be as above. Let $p \in M$ be any finite trace projection and $A \subseteq pMp$ be any von Neumann subalgebra. Let $\mathcal{K} = L^2(Mp) \otimes L^2(G)$ and view Φ as a normal $*$-morphism $\Phi : M \to B(\mathcal{K})$. Also, define the normal $*$-antimorphism $\rho : A \to B(\mathcal{K})$ given by $\rho(a)\xi = \xi(a \otimes 1)$ for all $a \in A$. Define $\mathcal{N} = \Phi(M) \vee \rho(A)$ as the von Neumann subalgebra of $B(\mathcal{K})$ generated by $\Phi(M)$ and $\rho(A)$. Denote by $\mathcal{N}_0 \subseteq \mathcal{N}$ the dense C*-subalgebra defined as $\mathcal{N}_0 := [\Phi(M)\rho(A)]$. Write $q = \Phi(p)$.

A normal completely bounded map $\psi : pMp \to pMp$ is called *adapted* with respect to A if the following two conditions hold.

(i) There exists a normal completely bounded map $\theta : q\mathcal{N}q \to B(L^2(pMp))$ with

$$\theta(\Phi(x)\rho(a)) = \psi(x)\rho(a) \tag{4.2.2}$$

for all $x \in pMp$ and $a \in A$.

(ii) There exist a Hilbert space $\mathcal{L}$, a unital $*$-homomorphism $\pi_0 : q\mathcal{N}_0 q \to B(\mathcal{L})$ and maps $\mathcal{V}, \mathcal{W} : \mathcal{N}_{pMp}(A) \to \mathcal{L}$ such that

$$\operatorname{Tr}(w^*\psi(x)va) = \langle \pi_0(\Phi(x)\rho(a))\mathcal{V}(v), \mathcal{W}(w)\rangle \tag{4.2.3}$$

for all $x \in pMp$, $a \in A$ and $v, w \in \mathcal{N}_{pMp}(A)$ and such that $\|\mathcal{V}\|_\infty \|\mathcal{W}\|_\infty < \infty$, where $\|\mathcal{V}\|_\infty = \sup\{\|\mathcal{V}(v)\| \mid v \in \mathcal{N}_{pMp}(A)\}$.

We denote by $\|\psi\|_{\text{adap}}$ the infimum of all possible values of $\operatorname{Tr}(p)^{-1}\|\mathcal{V}\|_\infty \|\mathcal{W}\|_\infty$.

We prove the following lemma in a slightly more general context than we will need below, since we will also use it in Section 4.5.

Lemma 4.2.7. *Let M, G, Φ be as above. Let $p \in M$ be any finite trace projection and $A \subseteq pMp$ be any von Neumann subalgebra. Suppose that $A \subseteq pMp$ is Φ-amenable. Suppose that $\omega \in A(G)$ and denote by $m : L(G) \to L(G)$ and $\varphi : M \to M$ be the associated normal completely bounded maps as in Lemma 4.2.3 and (4.2.1). Then, the map*

$$\psi : pMp \to pMp : x \mapsto p\varphi(x)p$$

is adapted with respect to A and $\|\psi\|_{adap} \leqslant \|m\|_{cb}$. Moreover, we can take the completely bounded map $\theta : q\mathcal{N}q \to B\big(L^2(pMp)\big)$ from Definition 4.2.6 such that $\|\theta\|_{cb} \leqslant \|m\|_{cb}$.

Proof. We use the notations from Definition 4.2.6. Denote $q = \Phi(p)$. To prove this lemma, we first prove that the pMp-A-bimodule ${}_{pMp}L^2(pMp)_A$ is weakly contained in the pMp-A-bimodule $\mathcal{L} = q(L^2(Mp) \otimes L^2(G))$ defined by $x \cdot \xi \cdot a = \Phi(x)\xi(a \otimes 1)$. We view $L^2(G) \otimes L^2(G)$ as the standard Hilbert space of $B\big(L^2(G)\big)$ in the following way. The von Neumann algebra $B\big(L^2(G)\big)$ is represented on $L^2(G) \otimes L^2(G)$ via $\pi_\ell(T) = T \otimes 1$. The modular conjugation J_0 is given by

$$J_0(\xi \otimes \eta) = \Sigma(\hat{J} \otimes \hat{J})(\xi \otimes \eta) = \overline{\eta} \otimes \overline{\xi}$$

for $\xi, \eta \in L^2(G)$. Here, $\Sigma : L^2(G) \otimes L^2(G) \to L^2(G) \otimes L^2(G)$ denotes the flip given by $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$ and $\hat{J} : L^2(G) \to L^2(G)$ denotes the anti-unitary operator given by $\hat{J}\xi = \overline{\xi}$. The right action is then given by

$$\pi_r(T)(\xi \otimes \eta) = J_0(T^* \otimes 1)J_0(\xi \otimes \eta) = \xi \otimes \hat{J}T^*\hat{J}\eta.$$

We can then view

$$\mathcal{K} = L^2(M) \otimes L^2(G) \otimes L^2(G)$$

as the standard Hilbert space for $M \mathbin{\overline{\otimes}} B(L^2(G))$. The left representation of $M \mathbin{\overline{\otimes}} B(L^2(G))$ on $\mathcal{K}$ is given by left multiplication in tensor positions 1 and 2. Denoting by $J : L^2(M) \to L^2(M)$ the modular conjugation from the standard representation of M on $L^2(M)$, we have that the modular conjugation $\widetilde{J}$ on $\mathcal{K}$ is given by $\widetilde{J} = J \otimes J_0$. The right anti-representation of $M \mathbin{\overline{\otimes}} B(L^2(G))$ on $\mathcal{K}$ is then given by

$$\xi \cdot T = \widetilde{J}(T^* \otimes 1)\widetilde{J}\xi$$

for $T \in M \mathbin{\overline{\otimes}} B(L^2(G))$. Denote $\mathcal{K}' = q \cdot \mathcal{K} \cdot q$ and view it as the standard Hilbert space of $q\big(M \mathbin{\overline{\otimes}} B(L^2(G))\big)$.

Let Ω be a $\Phi(A)$-central positive linear functional on $q\big(M \mathbin{\overline{\otimes}} B(L^2(G))\big)q$ as in the definition Φ-amenability. Approximating Ω in the weak* topology by a net $(\omega_i)_i$ of normal positive linear functionals, we get a net of vectors $(\xi_i)_i$ satisfying

$$\lim_i \langle (\Phi(x) \otimes 1)\xi_i, \xi_i \rangle = \lim_i \omega_i\big(\Phi(x)\big) = \mathrm{Tr}(x) \qquad (4.2.4)$$

for $x \in pMp$. Since Ω is $\Phi(A)$-central, also the net of functionals $\omega_i \circ \mathrm{Ad}\,\big(\Phi(u)\big)$ for $u \in \mathcal{U}(A)$ converges to Ω in weak* topology. Taking convex combinations, we can construct a net $\tilde{\omega}_i$ converging to Ω in weak* topology satisfying $\big\|\tilde{\omega}_i \circ \mathrm{Ad}\,\big(\Phi(u)\big) - \tilde{\omega}_i\big\| \to 0$ for all $u \in \mathcal{U}(A)$. Obviously, the net of vectors $(\tilde{\xi}_i)_i$ implementing $\tilde{\omega}_i$ still satisfies (4.2.4). Moreover, since $\omega_i \circ \mathrm{Ad}\,\big(\Phi(u)\big)$ is implemented by the vector $\Phi(u^*) \cdot \xi_i \cdot \Phi(u)$ (see also [Tak03a, Theorem 1.2. (iii)]) and using the Powers-Størmer inequality, we also find

$$\lim_i \big\|\Phi(u) \cdot \tilde{\xi}_i - \tilde{\xi}_i \cdot \Phi(u)\big\| = 0$$

for all $u \in \mathcal{U}(A)$. Since every element in a von Neumann algebra can be written as the sum of four unitaries, we have

$$\lim_i \left\|\Phi(a) \cdot \tilde{\xi}_i - \tilde{\xi}_i \cdot \Phi(a)\right\| = 0$$

for all $a \in A$. By Proposition 2.4.55, it then follows that the trivial pMp-A-bimodule $L^2(pMp)$ is weakly contained in the pMp-A-bimodule ${}_{\Phi(pMp)}\mathcal{K}'_{\Phi(A)}$. Denoting by V the canonical unitary implementation of the coaction Φ (see Theorem 4.1.4), we have

$$V(x \otimes 1)V^* = \Phi(x) \qquad \text{and} \qquad V(J \otimes \hat{J}) = (J \otimes \hat{J})V^*$$

for $x \in M$. Hence, using the leg numbering notation for tensor products, we have

$$V_{12}\Sigma_{23}\Phi(x)_{12}\Sigma_{23}V_{12}^* = V_{12}\Phi(x)_{13}V_{12}^* = (\Phi \otimes \mathrm{id})\big(\Phi(x)\big) = (\mathrm{id} \otimes \Delta)\big(\Phi(x)\big)$$

and

$$\begin{aligned} V_{12}\Sigma_{23}\widetilde{J}\Phi(x^*)_{12}\widetilde{J}\Sigma_{23}V_{12}^* &= V_{12}(J \otimes \hat{J} \otimes \hat{J})\Phi(x^*)_{12}(J \otimes \hat{J} \otimes \hat{J})V_{12}^* \\ &= (J \otimes \hat{J} \otimes \hat{J})V_{12}^*\Phi(x^*)_{12}V_{12}(J \otimes \hat{J} \otimes \hat{J}) \\ &= Jx^*J \otimes 1 \otimes 1 \end{aligned}$$

for all $x \in pMp$. Hence, the pMp-bimodule ${}_{\Phi(pMp)}\mathcal{K}'_{\Phi(pMp)}$ is unitarily conjugated with the pMp-bimodule ${}_{(\mathrm{id} \otimes \Delta)\Phi(pMp)}\mathcal{K}'_{M \otimes 1 \otimes 1}$. Now, using the unitary U from Example 4.1.2 satisfying $U(x \otimes 1)U^* = \Delta(x)$, we see that the pMp-bimodule ${}_{(\mathrm{id} \otimes \Delta)\Phi(pMp)}\mathcal{K}'_{pMp \otimes 1 \otimes 1}$ is unitarily conjugate to the pMp-bimodule ${}_{\Phi(pMp) \otimes 1}\mathcal{K}'_{pMp \otimes 1 \otimes 1}$, which is a multiple of the pMp-bimodule $\mathcal{L}$. Hence, the above weak containment statement is proven.

By Theorem 2.4.54, it follows that for all $x_1, \dots, x_n \in pMp$ and $a_1, \dots, a_n \in A$, we have

$$\left\|\sum_{i=1}^n x_i\rho(a_i)\right\| = \left\|\pi_{L^2(pMp)}(x)\right\| \leqslant \|\pi_{\mathcal{L}}(x)\| = \left\|\sum_{i=1}^n \Phi(x_i)\rho(a_i)\right\|,$$

where $x \in pMp \otimes_{\mathrm{alg}} A$ denotes the element $x = \sum_{i=1}^n x_i \otimes a_i$, and $\pi_{L^2(pMp)}$ and $\pi_{\mathcal{L}}$ denote the *-representations of $pMp \otimes_{\mathrm{alg}} A$ associated to the bimodules ${}_{pMp}L^2(pMp)_A$ and ${}_{pMp}\mathcal{L}_A$ respectively. Hence, the map $\Phi(x)\rho(a) \mapsto x\rho(a)$ for $x \in pMp$ and $a \in A$ extends to a unital $*$-morphism $\theta' : q\mathcal{N}_0q \to B\big(L^2(pMp)\big)$.

Now, define the normal completely bounded map

$$\theta : q\mathcal{N}q \to B(L^2(pMp)) : x \mapsto p(\mathrm{id} \otimes \omega)(x)p.$$

By construction, $\theta\big(\Phi(x)\rho(a)\big) = \psi(x)\rho(a)$ for all $x \in pMp$ and $a \in A$. Moreover, using Lemma 4.2.4, we have

$$\theta'\big(q(\mathrm{id}\otimes m)\big(\Phi(x)\rho(a)\big)q\big) = \theta'\big(q(\mathrm{id}\otimes m)\big(\Phi(x)\big)q\rho(a)\big) = \theta'\big(q\Phi\big(\varphi(x)\big)q\rho(a)\big)$$
$$= \theta'\big(\Phi\big(\psi(x)\big)\rho(a)\big) = \psi(x)\rho(a) = \theta\big(\Phi(x)\rho(a)\big)$$

for all $x \in pMp$ and $a \in A$ and hence $\theta(x) = \theta'\big(q(\mathrm{id}\otimes m)(x)q\big)$ for all $x \in q\mathcal{N}_0 q$. This implies that $\|\theta\|_{\mathrm{cb}} \leqslant \|m\|_{\mathrm{cb}}$. So, the Stinespring like factorization theorem (see e.g. [BO08, Theorem B.7]) provides a Hilbert space $\mathcal{L}'$, a unital $*$-homomorphism $\pi_0 : q\mathcal{N}_0 q \to B(\mathcal{L}')$ and bounded operators $\mathcal{V}_0, \mathcal{W}_0 : L^2(pMp) \to \mathcal{L}'$ satisfying $\theta(x) = \mathcal{W}_0^*\pi_0(x)\mathcal{V}_0$ for all $x \in q\mathcal{N}_0 q$ and $\|\mathcal{V}_0\|\,\|\mathcal{W}_0\| = \|\theta\|_{\mathrm{cb}} \leqslant \|m\|_{\mathrm{cb}}$. It now suffices to define $\mathcal{V}$ and $\mathcal{W}$ by restricting $\mathcal{V}_0$ and $\mathcal{W}_0$ to $\mathcal{N}_{pMp}(A) \subseteq L^2(pMp)$. So we have proved that $\psi : pMp \to pMp$ is adapted and that $\|\psi\|_{\mathrm{adap}} \leqslant \|m\|_{\mathrm{cb}}$. This concludes the proof of step 1. □

We now start the proof of the main theorem of this section.

Proof of Theorem 4.2.2. As before, let $p \in M$ be a finite trace projection and $A \subseteq pMp$ a von Neumann algebra that is Φ-amenable. Denote $\mathcal{K} = L^2(Mp)\otimes L^2(G)$ and view Φ as a normal $*$-homomorphism $\Phi : M \to B(\mathcal{K})$. Also define the normal $*$-antihomomorphism $\rho : A \to B(\mathcal{K})$ given by $\rho(a)\xi = \xi(a\otimes 1)$. Define $\mathcal{N} = \Phi(M) \vee \rho(A)$ as the von Neumann subalgebra of $B(\mathcal{K})$ generated by $\Phi(M)$ and $\rho(A)$. Note that $\mathcal{N} \subseteq B(L^2(Mp))\,\overline{\otimes}\,L(G)$. We also denote by $\rho : A \to B(L^2(Mp))$ the $*$-antihomomorphism given by right multiplication.

Whenever $\mathcal{V}$ is a set of operators on a Hilbert space, we denote by $[\mathcal{V}]$ the operator norm closed linear span of $\mathcal{V}$. Denote by $\mathcal{N}_0 \subseteq \mathcal{N}$ the dense C*-subalgebra defined as $\mathcal{N}_0 := [\Phi(M)\rho(A)]$. Write $q = \Phi(p)$.

Let $(\eta_i)_i$ be a net in $A_c(G)$ as in Lemma 4.2.5. Let $m_i : L(G) \to L(G)$ and $\varphi_i : M \to M$ be the associated normal completely bounded maps.

By Lemma 4.2.7, we obtain a net of completely bounded maps $\psi_i : pMp \to pMp$ that are adapted with respect to A and such that $\psi_i(x) \to x$ in s.o. topology for all $x \in pMp$ and $\limsup_n \|\psi_i\|_{\mathrm{adap}} < \infty$ for all n. We call such a net an *adapted approximate identity*. We then define $\kappa \geqslant 0$ as the infimum of all positive numbers L for which there exists an adapted approximate identity $\psi_i : pMp \to pMp$ with $\limsup_i \|\psi_i\|_{\mathrm{adap}} \leqslant L$. We fix such a ψ_i with $\lim_i \|\psi_i\|_{\mathrm{adap}} = \kappa$.

Since each ψ_i is adapted, we have normal completely bounded maps $\theta_i : q\mathcal{N}q \to B\big(L^2(pMp)\big)$, Hilbert spaces $\mathcal{L}_i$ unital $*$-morphisms $\pi_i : q\mathcal{N}_0 q \to B(\mathcal{L}_i)$ and maps $\mathcal{V}_i, \mathcal{W}_i : \mathcal{N}_{pMp}(A) \to \mathcal{L}$ satisfying (4.2.2) and (4.2.3). Moreover, we can assume that $\lim_i \|\mathcal{V}_i\|_\infty = \sqrt{\kappa}$ and $\lim_i \|\mathcal{W}_i\|_\infty = \mathrm{Tr}(p)\sqrt{\kappa}$.

For every $v \in \mathcal{N}_{pMp}(A)$, denote by β_v the automorphism of $\mathcal{N}$ implemented by right multiplication with $v^* \otimes 1$ on $L^2(Mp) \otimes L^2(G)$. Note that $\beta_v(\Phi(x)\rho(a)) = \Phi(x)\rho(vav^*)$ for all $x \in M$ and $a \in A$. In particular, $\beta_v(q) = q$ and we also view β_v as an automorphism of $q\mathcal{N}q$.

Recall that for any linear functional $\mu : q\mathcal{N}q \to \mathbb{C}$ and any $v \in q\mathcal{N}q$, we denote by $v \cdot \mu$ the linear functional given by $(v \cdot \mu)(x) = \mu(xv)$.

Step 1. There exist normal linear functionals $\mu_i : q\mathcal{N}q \to \mathbb{C}$ satisfying the following properties.

(a) $\limsup_i \|\mu_i\| < \infty$,

(b) $\lim_i \mu_i\big(\Phi(x)\rho(a)\big) = \mathrm{Tr}(xa)$ for all $x \in pMp$, $a \in A$,

(c) $\lim_i \|\mu_i \circ \beta_u \circ \mathrm{Ad}\,\Phi(u) - \mu_i\| = 0$ for all $u \in \mathcal{N}_{pMp}(A)$,

(d) $\lim_i \|\big(\Phi(v^*)\rho(v)\big) \cdot \mu_i - \mu_i\| = 0$ for all $v \in \mathcal{U}(A)$.

The proof of step 1 is the same as the argument in [Oza12, Proof of Proposition 7] and [PV14a, Proof of Proposition 5.4]. For completeness, we write out a full proof here.

We define $\mu_i \in (q\mathcal{N}q)_*$ by $\mu_i(T) = \langle \theta_i(T)\hat{p}, \hat{p}\rangle$. Note that $\mu_i\big(\Phi(x)\rho(a)\big) = \mathrm{Tr}\,\big(\psi_i(x)a\big)$ and hence,

$$\lim_i \mu_i\big(\Phi(x)\rho(a)\big) = \lim_i \mathrm{Tr}\,\big(\psi_i(x)a\big) = \mathrm{Tr}(xa)$$

for all $x \in pMp$ and $a \in A$. Moreover, we also have

$$|\mu_i(x)| = |\,\langle \pi_i(x)\mathcal{V}_i(p), \mathcal{W}_i(p)\rangle\,| \leqslant \|x\|\,\|\mathcal{V}_i(p)\|\,\|\mathcal{W}_i(p)\|$$

for all $x \in \mathrm{span}\{\Phi(x)\rho(a)\}_{x\in pMp, a\in A}$. Since this linear span is a w.o. dense *-subalgebra of $q\mathcal{N}q$, it follows that $\|\mu_i\| \leqslant \|\mathcal{V}_i\|_\infty \|\mathcal{W}_i\|_\infty$ and hence $\limsup_i \|\mu_i\| = \mathrm{Tr}(p)\kappa < \infty$.

Fix a $u \in \mathcal{N}_{pMp}(A)$ and define maps $\psi_i^u : pMp \to pMp$ by $\psi_i^u(x) = \psi_i(xu^*)u$. One checks that the net $(\psi_i^u)_i$ also forms an adapted approximate identity. Indeed, one checks that the map

$$\theta_i^u : q\mathcal{N}q \to B\big(L^2(pMp)\big) : x \mapsto \theta_i\big(x\Phi(u)^*\big)u$$

satisfies (4.2.2) and that the map

$$\mathcal{V}_i^u : \mathcal{N}_{pMp}(A) \to \mathcal{L} : v \mapsto \pi_i\big(\Phi(u)\big)^*\mathcal{V}_i(uv)$$

together with the maps π_i and $\mathcal{W}_i$ above satisfies (4.2.3).

Clearly, then also the net $\frac{1}{2}(\phi_i + \phi_i^u)$ is an adapted approximate identity, where the maps satisfying (4.2.2) and (4.2.3) are given by $\frac{1}{2}(\theta_i + \theta_i^u)$, π_i, $\frac{1}{2}(\mathcal{V}_i + \mathcal{V}_i^u)$ and $\mathcal{W}_i$. Since κ was minimal, we have

$$\begin{aligned}\sqrt{\kappa}\liminf_i \left\|\tfrac{1}{2}(\mathcal{V}_i + \mathcal{V}_i^u)\right\|_\infty &= \liminf_i \operatorname{Tr}(p)^{-1}\left\|\tfrac{1}{2}(\mathcal{V}_i + \mathcal{V}_i^u)\right\|_\infty \|\mathcal{W}_i\|_\infty \\ &\geqslant \liminf_i \left\|\tfrac{1}{2}(\psi_i + \psi_i^u)\right\|_{adap} \\ &\geqslant \kappa.\end{aligned}$$

So, we can take a net $v_i \in \mathcal{N}_{pMp}(A)$ such that

$$\liminf_i \left\|\tfrac{1}{2}\big(\mathcal{V}_i(v_i) + \mathcal{V}_i^u(v_i)\big)\right\| \geqslant \sqrt{\kappa}.$$

Since

$$\lim_i \|\mathcal{V}_i(v_i)\| \leqslant \lim_i \|\mathcal{V}_i\|_\infty = \sqrt{\kappa}$$

and

$$\lim_i \|\mathcal{V}_i^u(v_i)\| \leqslant \lim_i \|\mathcal{V}_i^u\|_\infty = \lim_i \|\mathcal{V}_i\|_\infty = \sqrt{\kappa},$$

applying the parallelogram law yields that $\|\mathcal{V}_i(v_i) - \mathcal{V}_i^u(v_i)\| \to 0$.

Now, define $\mu_i^u \in (q\mathcal{N}q)_*$ by $\mu(T) = \langle \theta_i^u(T)\hat{p}, \hat{p}\rangle$. As before, note that $\mu_i^u\big(\Phi(x)\rho(a)\big) = \operatorname{Tr}\big(\psi_i^u(x)a\big)$. We have

$$\begin{aligned}(\mu_i^u \circ \beta_{v_i})\big(\Phi(x)\rho(a)\big) &= \mu_i^u\big(\Phi(x)\rho(v_i a v_i^*)\big) = \operatorname{Tr}\big(\phi_i^u(x) v_i a v_i^*\big) \\ &= \big\langle \pi_i\big(\Phi(x)\rho(a)\big)\mathcal{V}_i^u(v_i), \mathcal{W}_i(v_i)\big\rangle\end{aligned}$$

and

$$\begin{aligned}(\mu_i \circ \beta_{v_i})\big(\Phi(x)\rho(a)\big) &= \mu_i\big(\Phi(x)\rho(v_i a v_i^*)\big) = \operatorname{Tr}\big(\phi_i(x) v_i a v_i^*\big) \\ &= \big\langle \pi_i\big(\Phi(x)\rho(a)\big)\mathcal{V}_i(v_i), \mathcal{W}_i(v_i)\big\rangle\end{aligned}$$

for all $x \in pMp$ and $a \in A$. Since $\operatorname{span}\{\Phi(x)\rho(a)\}_{x\in pMp, a\in A}$ is a w.o. dense $*$-subalgebra of $q\mathcal{N}q$, it follows from the Kaplansky density theorem that

$$\lim_i \|\mu_i - \mu_i^u\| = \|(\mu_i - \mu_i^u)\circ\beta_{v_i}\| \leqslant \lim_i \|\mathcal{V}_i^u(v_i) - \mathcal{V}_i(v_i)\|\,\|\mathcal{W}_i(v_i)\| = 0.$$

Similarly, the maps ${}^u\psi_i^u : pMp \to pMp$ defined by

$${}^u\psi_i^u(x) = u^*\psi_i^u(ux) = u^*\psi_i(uxu^*)u$$

forms an adapted approximate identity, where the maps ${}^u\theta_i^u$, π_i, $\mathcal{V}_i^u$ and $\mathcal{W}_i^u$ satisfying (4.2.2) and (4.2.3) are given by

$$ {}^u\theta_i^u(x) = u^*\theta_i(\Phi(u)x\Phi(u)^*)u $$

for $x \in q\mathcal{N}q$,

$$ \mathcal{W}_i^u(w) = \pi_i\big(\Phi(u)\big)^*\mathcal{W}_i(uw) $$

for $w \in \mathcal{N}_{pMp}(A)$, and π_i and V_i^u are as before. Similarly as above, we find that for the associated functional ${}^u\mu_i^u \in (q\mathcal{N}q)_*$, defined by ${}^u\mu_i^u(T) = \langle {}^u\theta_i^u(T)\hat{p}, \hat{p}\rangle$ for $T \in q\mathcal{N}q$, we have

$$ \lim_i \|{}^u\mu_i^u - \mu_i^u\| = \lim_i \|({}^u\mu_i^u - \mu_i^u) \circ \beta_{v_i}\| = 0. $$

Now, note that

$$ {}^u\mu_i^u\big(\Phi(x)\rho(a)\big) = \mathrm{Tr}\left({}^u\psi^u(x)a\right) = \mathrm{Tr}\left(\psi_i(uxu^*)uau^*\right) = \mu_i\big(\Phi(uxu^*)\rho(uau^*)\big) $$

for all $x \in pMp$ and $a \in A$. Hence, ${}^u\mu_i^u = \mu_i \circ \beta_u \circ \mathrm{Ad}\,\Phi(u)$ and we have proven (c).

Finally, we prove (d). Fix $v \in \mathcal{U}(A)$. For every $x \in pMp$ and $a \in A$, we have

$$ \mu_i^v\big(\Phi(x)\rho(a)\big) = \mathrm{Tr}\left(\psi(xv^*)va\right) = \mu_i\big(\Phi(xv^*)\rho(va)\big) = \mu_i\big(\Phi(x)\rho(a)\Phi(v)^*\rho(v)\big), $$

where we used that ρ is a $*$-antimorphism in the last step. Hence, $\mu_i^v = \big(\Phi(v^*)\rho(v)\big) \cdot \mu_i$. Since $v \in \mathcal{U}(A) \subseteq \mathcal{N}_{pMp}(A)$, we have from the previous that $\|\mu_i^v - \mu_i\| \to 0$ which proves step 1.

Step 2. There exist a net of *positive* normal linear functionals $\omega_i \in (q\mathcal{N}q)_*$ satisfying

(a) $\lim_i \omega_i\big(\Phi(x)\big) = \mathrm{Tr}(x)$ for all $x \in pMp$,

(b) $\lim_i \|\omega_i \circ \beta_u \circ \mathrm{Ad}\,\Phi(u) - \omega_i\| = 0$ for all $u \in \mathcal{N}_{pMp}(A)$,

(c) $\lim_i \omega_i\big(\Phi(v)\rho(v^*)\big) = \mathrm{Tr}(p)$ for all $v \in \mathcal{U}(A)$.

To prove step 2, choose a weak* limit point $\Psi \in (q\mathcal{N}q)^*$ of the net μ_i. We find that $\Psi(\Phi(x)) = \mathrm{Tr}(x)$ for all $x \in pMp$, that Ψ is invariant under the automorphisms $\beta_u \circ \mathrm{Ad}\,\Phi(u)$ for all $u \in \mathcal{N}_{pMp}(A)$ and that $\big(\Phi(v)\rho(v^*)\big) \cdot \Psi = \Psi$ for all $v \in \mathcal{U}(A)$. Set $\Psi_1 = |\Psi| \in (q\mathcal{N}q)^*_+$ (see Theorem 4.1.11). By Theorem 4.1.12, we have

$$ \Psi_1 \circ (\beta_u \circ \mathrm{Ad}\,\Phi(u)) = \Psi_1 \quad \text{and} \quad (\Phi(v)\rho(v^*)) \cdot \Psi_1 = \Psi_1 \tag{4.2.5} $$

for $u \in \mathcal{N}_{pMp}(A)$ and $v \in \mathcal{U}(A)$. Furthermore, taking a unitary $u \in \big(q\mathcal{N}q)^{**}$ such that $\Psi(x) = \Psi_1(xu)$, we find

$$|\operatorname{Tr}(x)|^2 = |\Psi\big(\Phi(x)\big)|^2 = |\Psi_1\big(\Phi(x)u\big)| \leqslant \|\Psi_1\|\Psi_1\big(\Phi(x^*x)\big)$$

for all $x \in pMp$. In order to conclude the proof of step 2, we need to modify Ψ_1 so that its restriction to $\Phi(pMp)$ is given by the trace. We first modify Ψ_1 so that this restriction is normal and faithful.

Consider the second duals $(pMp)^{**}$ and $(q\mathcal{N}q)^{**}$ of the von Neumann algebras pMp and $q\mathcal{N}q$ as in Section 4.1.3. By Theorem 4.1.10, the embedding $\Phi : pMp \hookrightarrow q\mathcal{N}q$ yields an embedding $\Phi^{**} : (pMp)^{**} \hookrightarrow (q\mathcal{N}q)^{**}$. Denote by $z \in \mathcal{Z}\big((pMp)^{**}\big)$ the support projection of the natural normal $*$-homomorphism $(pMp)^{**} \to pMp$. Write $z_1 = \Phi^{**}(z) \in \mathcal{Z}\big(\Phi(pMp)^{**}\big)$ and note that z_1 is the support projection of the natural normal $*$-morphism $\theta : \big(\Phi(pMp)\big)^{**} \to \Phi(pMp)$. Whenever $\alpha \in \operatorname{Aut}(q\mathcal{N}q)$ satisfies $\alpha\big(\Phi(pMp)\big) = \Phi(pMp)$, the bidual automorphism $\alpha^{**} \in \operatorname{Aut}\big((q\mathcal{N}q)^{**}\big)$ satisfies $\alpha^{**}\big((pMp)^{**}\big) = (pMp)^{**}$. Since $\theta \circ \alpha^{**} = \alpha \circ \theta$, we have that also $\alpha^{**}(z_1)$ is the support projection of Ψ and hence $\alpha^{**}(z_1) = z_1$. In particular, z_1 commutes with every unitary in $q\mathcal{N}q$ that normalizes $\Phi(pMp)$. Applying this to the automorphism $\alpha = \beta_u \circ \operatorname{Ad}\Phi(u)$ for $u \in \mathcal{N}_{pMp}(A)$ and the unitary $\Phi(v)\rho(v^*)$ for $v \in \mathcal{U}(A)$, it follows that the positive functional

$$\Psi_2 : q\mathcal{N}q \to \mathbb{C} : x \mapsto \Psi_1(xz_1)$$

still satisfies the properties in (4.2.5) and Ψ_2 is normal by Theorem 4.1.9.

By density of pMp in $(pMp)^{**}$, we have $|\operatorname{Tr}(x)|^2 \leqslant \|\Psi_1\|\,\Psi_1\big(\Phi^{**}(x^*x)\big)$ for all $x \in (pMp)^{**}$. Since $\operatorname{Tr}(x) = \operatorname{Tr}(xz)$ (see Theorem 4.1.9) for all $x \in pMp$, we conclude that $|\operatorname{Tr}(x)|^2 \leqslant \|\Psi_1\|\,\Psi_2\big(\Phi(x^*x)\big)$ for all $x \in pMp$. In particular, $\Psi_2 \circ \Phi$ is faithful. Since $\Psi_2(\Phi(x)) = \Psi_2(\Phi^{**}(xz))$ for all $x \in pMp$, we get that $\Psi_2 \circ \Phi$ is normal. So, we find a $T \in L^1(pMp)^+$ with trivial kernel such that $\Psi_2(\Phi(x)) = \operatorname{Tr}(xT)$ for all $x \in pMp$. Since (4.2.5) holds and $\beta_u(\Phi(x)) = \Phi(x)$ for all $x \in pMp$, we have that T commutes with $\mathcal{N}_{pMp}(A)$. For every $n \geqslant 3$, we then define the positive functional Ψ_n on $q\mathcal{N}q$ given by

$$\Psi_n(x) = \Psi_2\left(\Phi\left(\left(T + \tfrac{1}{n}\right)^{-1/2}\right) x \Phi\left(\left(T + \tfrac{1}{n}\right)^{-1/2}\right)\right)$$

for $x \in pMp$. Each Ψ_n satisfies the properties in (4.2.5). Choosing Ψ to be a weak*-limit point of the sequence Ψ_n, we have found a positive functional Ψ on $q\mathcal{N}q$ that satisfies the properties in (4.2.5) and that moreover satisfies $\Psi(\Phi(x)) = \operatorname{Tr}(x)$ for all $x \in pMp$. Approximating Ψ in the weak* topology and taking convex combinations, we find a net of positive $\omega_n \in (q\mathcal{N}q)_*$ satisfying the requirements of step 2.

Notations and terminology. Choose a standard Hilbert space $\mathcal{H}$ for the von Neumann algebra $\mathcal{N}$, which comes with the normal $*$-homomorphism $\pi_\ell : \mathcal{N} \to B(\mathcal{H})$, the normal $*$-antihomomorphism $\pi_r : \mathcal{N} \to B(\mathcal{H})$ and the positive cone $P \subseteq \mathcal{H}$. For every $v \in \mathcal{N}_{pMp}(A)$, denote by $W_v \in \mathcal{U}(\mathcal{H})$ the canonical implementation of $\beta_v \in \mathrm{Aut}(\mathcal{N})$.

Step 3. There exist a net $(\xi_i)_i$ of vectors in P satisfying $\pi_\ell(q)\xi_i = \xi_i = \pi_r(q)\xi_i$ for all i and

(a) $\lim_i \langle \pi_\ell(\Phi(x))\xi_i, \xi_i \rangle = \mathrm{Tr}(pxp) = \lim_i \langle \pi_r(\Phi(x))\xi_i, \xi_i \rangle$ for all $x \in M$,

(b) $\lim_i \|\pi_\ell(\Phi(u))\pi_r(\Phi(u^*))W_u\xi_i - \xi_i\| = 0$ for all $u \in \mathcal{N}_{pMp}(A)$,

(c) $\lim_i \|\pi_\ell(\Phi(v))\xi_i - \pi_\ell(\rho(v))\xi_i\| = 0$ for all $v \in \mathcal{U}(A)$.

Note that $\pi_\ell(q)\pi_r(q)\mathcal{H}$ serves as the standard Hilbert space of $q\mathcal{N}q$. Let $\xi_i \in \pi_\ell(q)\pi_r(q)P$ be the vector implementing the normal positive linear functional ω_i. Clearly, the first property of step 2 translates into the first property of this step. Also, the third property of step 2 implies the third property of this step. Indeed,

$$\begin{aligned}\left\|\pi_\ell(\Phi(v))\xi_i - \pi_\ell(\rho(v))\xi_i\right\|^2 &= 2\|\xi_i\|^2 - 2\,\mathrm{Re}\left(\langle \pi_\ell(\rho(v^*)\Phi(v))\xi_i, \xi_i \rangle\right) \\ &= 2\omega_i(q) - 2\omega_i(\Phi(v)\rho(v^*)) \\ &\to 0\end{aligned}$$

To prove the second property, note that each functional $\omega_i \circ \beta_u \circ \mathrm{Ad}(\Phi(u))$ for $u \in \mathcal{N}_{pMp}(A)$ is implemented by the vector $\pi_\ell(\Phi(u))\pi_r(\Phi(u^*))W_u\xi_i \in \pi_\ell(q)\pi_r(q)P$. Hence, by the Powers-Størmer inequality, we have

$$\|\pi_\ell(\Phi(u))\pi_r(\Phi(u^*))W_u\xi_i - \xi_i\|^2 \leqslant \|\omega_i \circ \beta_u \circ \mathrm{Ad}\,\Phi(u) - \omega_i\| \to 0$$

for all $u \in \mathcal{N}_{pMp}(A)$.

Notations and terminology. Define the coaction $\Psi : \mathcal{N} \to \mathcal{N} \mathbin{\overline{\otimes}} L(G)$ given by $\Psi = \mathrm{id} \otimes \Delta$. By Theorem 4.1.4, the coaction Ψ has a canonical unitary implementation $V \in B(\mathcal{H}) \mathbin{\overline{\otimes}} L(G)$. Let $\pi : C_0(G) \to B(\mathcal{H})$ be the associated $*$-morphism.

Formulation of the dichotomy. We are in precisely one of the following cases.

- **Case 1.** For every $F \in C_0(G)$, we have that $\limsup_i \|\pi(F)\xi_i\| = 0$.

- **Case 2.** There exists an $F \in C_0(G)$ with $\limsup_i \|\pi(F)\xi_i\| > 0$.

We prove that in case 1, the von Neumann subalgebra $\mathcal{N}_{pMp}(A)'' \subseteq pMp$ is Φ-amenable and that in case 2, the von Neumann subalgebra $A \subseteq pMp$ can be Φ-embedded.

Case 1 – Notations and terminology. Since G is in class $\mathcal{S}$, we can take a continuous map $\zeta : G \to L^2(G)$ as in Proposition 3.1.2 (iii). Define the operator

$$Z_0 : L^2(G) \mapsto L^2(G) \otimes L^2(G) : (Z_0\xi)(s,t) = \zeta(s)(t)\xi(s).$$

As before, we identify a vector $\xi \in L^2(G)$ with the operator

$$\xi : \mathbb{C} \to \mathcal{H} : \lambda \mapsto \lambda\xi.$$

Denoting by $C_0(G) \otimes_{\min} L^2(G) \subseteq B\big(L^2(G), L^2(G) \otimes L^2(G)\big)$ the norm closure of the linear span of the operators $f \otimes \xi$ for $f \in C_0(G)$ and $\xi \in L^2(G)$, we have that $Z_0 \in C_0(G) \otimes_{\min} L^2(G)$.

Define $\widetilde{V} \in B\big(L^2(G) \otimes L^2(G)\big)$ by

$$(\widetilde{V}\xi)(g,h) = \xi(h^{-1}g, h)$$

for $\xi \in L^2(G) \otimes L^2(G)$. Note that $\widetilde{V} \in M\big(C^*_\lambda(G) \otimes_{\min} C_0(G)\big)$. Moreover, $\Delta(x) = \widetilde{V}(1 \otimes x)\widetilde{V}^*$ for all $x \in L(G)$.

By [BSV03, Section 5], the closed linear span

$$M_0 = [(\mathrm{id} \otimes \omega)\big(\Phi(x)\big) \mid x \in M, \omega \in L(G)_*] \tag{4.2.6}$$

is a unital C*-subalgebra of M. Also, $\Phi(M_0) \subseteq M(M_0 \otimes_{\min} C^*_\lambda(G))$ and the restriction of Φ to M_0 defines a continuous coaction. In particular, the closed linear span

$$S_\ell = [\Phi(M_0)(1 \otimes_{\min} C_0(G))] \subseteq M \mathbin{\overline{\otimes}} B(L^2(G)) \tag{4.2.7}$$

is a C*-algebra (i.e. the crossed product of M_0 and the coaction Φ of $C^*_\lambda(G)$, as first defined in [BS93, Définition 7.1]) and

$$M_0 = [(\mathrm{id} \otimes \omega)\Phi(x) \mid x \in M_0, \omega \in L(G)_*]\,. \tag{4.2.8}$$

Case 1 – Step 1. We claim that

$$(1 \otimes Z_0)\Phi(x) - (\Phi \otimes \mathrm{id})\big(\Phi(x)\big)(1 \otimes Z_0) \in S_\ell \otimes_{\min} L^2(G) \tag{4.2.9}$$

for all $x \in M_0$.

Fix $F \in C_0(G)$. We have

$$\begin{aligned}\big(\widetilde{V}_{23}^*(Z_0 \otimes F) - \widetilde{V}_{13}(Z_0 \otimes F)\widetilde{V}^*\xi\big)(s,t,u) \\ &= \big((Z_0 \otimes F)\xi\big)(s,ut,u) - \big((Z_0 \otimes F)\widetilde{V}^*\xi\big)(u^{-1}s,t,u) \\ &= \big(\zeta(s)(ut) - \zeta(u^{-1}s)(t)\big)F(u)\xi(s,u)\end{aligned}$$

for every $\xi \in L^2(G) \otimes L^2(G)$ and a.e. $s,t,u \in G$. Since

$$\lim_{s\to\infty} \big\|\zeta(u^{-1}s) - \lambda_u^*\zeta(s)\big\|_2 = 0$$

uniformly on compact sets for $u \in G$, it follows that

$$\widetilde{V}_{23}^*(Z_0 \otimes F) - \widetilde{V}_{13}(Z_0 \otimes F)\widetilde{V}^* \in C_0(G) \otimes_{\min} L^2(G) \otimes_{\min} C_0(G).$$

Hence,

$$\widetilde{V}_{13}^*\widetilde{V}_{23}^*(Z_0 \otimes F) - (Z_0 \otimes F)\widetilde{V}^* \in \widetilde{V}_{13}^*\big(C_0(G) \otimes_{\min} L^2(G) \otimes_{\min} C_0(G)\big).$$

and

$$(Z_0 \otimes F) - V_{23}V_{13}(Z_0 \otimes F)\widetilde{V}^* \in \widetilde{V}_{23}\big(C_0(G) \otimes_{\min} L^2(G) \otimes_{\min} C_0(G)\big).$$

Since $\widetilde{V}$ normalizes $C_0(G) \otimes_{\min} C_0(G)$, it follows that

$$\widetilde{V}_{13}^*\widetilde{V}_{23}^*(Z_0 \otimes F) - (Z_0 \otimes F)\widetilde{V}^* \in \big(C_0(G) \otimes_{\min} L^2(G) \otimes_{\min} C_0(G)\big)\widetilde{V}^* \quad (4.2.10)$$

and since $\widetilde{V}\big(L^2(G) \otimes_{\min} C_0(G)\big) = L^2(G) \otimes_{\min} C_0(G)$, we similarly have that

$$\widetilde{V}_{23}\widetilde{V}_{13}(Z_0 \otimes F) - (Z_0 \otimes F)\widetilde{V} \in \big(C_0(G) \otimes_{\min} L^2(G) \otimes_{\min} C_0(G)\big)\widetilde{V}. \quad (4.2.11)$$

Recall that given $\xi, \eta \in L^2(G)$, we denote by $\omega_{\xi,\eta} \in L(G)_*$ the linear functional defined by $\omega_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$. By Theorem 2.4.26, it follows that the linear functionals $\omega_{\mu,\eta}$ for $\mu, \eta \in C_c(G)$ are $\|.\|$-dense in $L(G)_*$. Hence, by (4.2.8), it suffices to prove (4.2.9) for

$$x = (1 \otimes \eta^*)\big(\Phi(y)\big)(1 \otimes \mu) = (\mathrm{id} \otimes \omega_{\mu,\eta})\big(\Phi(y)\big)$$

where $y \in M_0$ and where $\eta, \mu \in C_c(G)$ are viewed as vectors in the Hilbert space $L^2(G)$. Let $F \in C_0(G)$ be such that $\eta^*F = \eta^*$ and $F\mu = \mu$. Using that Φ is a coaction and that $\Delta(a) = \widetilde{V}(1 \otimes a)\widetilde{V}^*$ for all $a \in L(G)$, we find that

$$\Phi(x) = \big((\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\mu,\eta}) \circ (\Phi \otimes \mathrm{id}) \circ \Phi\big)(y)$$

$$= \big((\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\mu,\eta}) \circ (\mathrm{id} \otimes \Delta) \circ \Phi\big)(y)$$

$$= (\mathrm{id} \otimes \mathrm{id} \otimes \omega_{\mu,\eta})\big(\widetilde{V}_{23}\Phi(y)_{13}\widetilde{V}_{23}^*\big)$$

and hence

$$\begin{aligned}(\Phi \otimes \mathrm{id})\big(\Phi(x)\big) &= \big((\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega_{\mu,\eta}) \circ (\Phi \otimes \mathrm{id} \otimes \mathrm{id})\big)\big(\widetilde{V}_{23}\Phi(y)_{13}\widetilde{V}_{23}^*\big) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega_{\mu,\eta})\big(\widetilde{V}_{34}\big((\Phi \otimes \mathrm{id})\Phi(y)\big)_{124}\widetilde{V}_{34}^*\big) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega_{\mu,\eta})\big(\widetilde{V}_{34}\big((\mathrm{id} \otimes \Delta)\Phi(y)\big)_{124}\widetilde{V}_{34}^*\big) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \omega_{\mu,\eta})\big(\widetilde{V}_{34}\widetilde{V}_{24}\Phi(y)_{14}\widetilde{V}_{24}^*\widetilde{V}_{34}^*\big) \\ &= (1 \otimes 1 \otimes 1 \otimes \eta^*)\widetilde{V}_{34}\widetilde{V}_{24}\Phi(y)_{14}\widetilde{V}_{24}^*\widetilde{V}_{34}^*(1 \otimes 1 \otimes 1 \otimes \mu)\end{aligned}$$

Twice using that $\mu = F\mu$, it then follows from (4.2.10) that

$$\begin{aligned}(\Phi \otimes \mathrm{id})\big(\Phi(x)\big)(1 \otimes Z_0) &= \eta_4^*\, \widetilde{V}_{34}\, \widetilde{V}_{24}\, \Phi(y)_{14}\, (1 \otimes Z_0 \otimes F)\, \widetilde{V}_{23}^*\, \mu_3 + T \\ &= \eta_4^*\, \widetilde{V}_{34}\, \widetilde{V}_{24}\, \Phi(y)_{14}\, (1 \otimes Z_0 \otimes 1)\, \widetilde{V}_{23}^*\, \mu_3 + T \\ &= \eta_4^*\, \widetilde{V}_{34}\, \widetilde{V}_{24}\, (1 \otimes Z_0 \otimes 1)\, \Phi(y)_{13}\, \widetilde{V}_{23}^*\, \mu_3 + T \qquad (4.2.12)\end{aligned}$$

where the error term T belongs to

$$\big[L^2(G)_4^*\, \widetilde{V}_{34}\, \widetilde{V}_{24}\, \Phi(M_0)_{14}\, \big(1 \otimes C_0(G) \otimes L^2(G) \otimes C_0(G)\big)\, \widetilde{V}_{23}^*\, L^2(G)_3\big].$$

Here, we write $\otimes$ instead of $\otimes_{\min}$ for brevity. Using that $\big[\Phi(M_0)\big(1 \otimes C_0(G)\big)\big] = S_\ell = \big[\big(1 \otimes C_0(G)\big)\Phi(M_0)\big]$, that $\widetilde{V}_{34}$ and $\widetilde{V}_{24}$ commute, that $\big[\widetilde{V}\big(L^2(G) \otimes C_0(G)\big)\big] = [L^2(G) \otimes C_0(G)]$ and $\widetilde{V}$ normalizes $C_0(G) \otimes_{\min} C_0(G) = C_0(G \times G)$ and that

$$M_0 = [(\mathrm{id} \otimes \omega)\Phi(x) \mid x \in M, \omega \in L(G)_*] = \big[\big(1 \otimes L(G)\big)^*\Phi(M_0)\big(1 \otimes L^2(G)\big)\big],$$

we get that T belongs to

$$\begin{aligned}&\big[L^2(G)_4^*\, \widetilde{V}_{34}\, \widetilde{V}_{24}\, \big(1 \otimes C_0(G) \otimes L^2(G) \otimes C_0(G)\big)\, \Phi(M_0)_{13}\, \widetilde{V}_{23}^*\, L^2(G)_3\big] \\ &= \big[L^2(G)_4^*\, \widetilde{V}_{24}\, \big(1 \otimes C_0(G) \otimes L^2(G) \otimes C_0(G)\big)\, \Phi(M_0)_{13}\, \widetilde{V}_{23}^*\, L^2(G)_3\big] \\ &= \big[\big(1 \otimes C_0(G) \otimes L^2(G)\big)\, L^2(G)_3^*\, \widetilde{V}_{23}\, \Phi(M_0)_{13}\, \widetilde{V}_{23}^*\, L^2(G)_3\big] \\ &= \big[\big(1 \otimes C_0(G) \otimes L^2(G)\big)\, L^2(G)_3^*\, (\Phi \otimes \mathrm{id})\big(\Phi(M_0)\big)\, L^2(G)_3\big]\end{aligned}$$

$$= \big[\big(1 \otimes C_0(G) \otimes L^2(G)\big)\,(\Phi \otimes \mathrm{id})\big((1 \otimes L(G))^* \Phi(M_0)(1 \otimes L^2(G))\big)\big]$$

$$= \big[\big(1 \otimes C_0(G)\big)\Phi(M_0) \otimes L^2(G)\big]$$

$$= S_\ell \otimes_{\min} L^2(G).$$

Using that $\eta^* = \eta^* F$ and using (4.2.11), we can continue the computation in (4.2.12) and find that

$$\begin{aligned}(\Phi \otimes \mathrm{id})\big(\Phi(x)\big)(1 \otimes Z_0) &= \eta_4^*\, \widetilde{V}_{34}\, \widetilde{V}_{24}\, (1 \otimes Z_0 \otimes F)\, \Phi(y)_{13}\, \widetilde{V}_{23}^*\, \mu_3 + T \\ &= \eta_4^*\, (1 \otimes Z_0 \otimes F)\, \widetilde{V}_{23}\, \Phi(y)_{13}\, \widetilde{V}_{23}^*\, \mu_3 + T' + T \\ &= (1 \otimes Z_0)\, \eta_3^*\, (\Phi \otimes \mathrm{id})\big(\Phi(y)\big)\, \mu_3 + T' + T \\ &= (1 \otimes Z_0)\, \Phi(x) + T' + T\end{aligned}$$

where the error term T' belongs to

$$\big[L^2(G)_4^*\, \big(1 \otimes C_0(G) \otimes L^2(G) \otimes C_0(G)\big)\, \widetilde{V}_{23}\, \Phi(M_0)_{13}\, \widetilde{V}_{23}^*\, L^2(G)_3\big]$$

$$= \big[\big(1 \otimes C_0(G) \otimes L^2(G)\big)\, L^2(G)_3^*\, (\Phi \otimes \mathrm{id})\big(\Phi(M_0)\big)\, L^2(G)_3\big]$$

$$= S_\ell \otimes_{\min} L^2(G)\ .$$

So (4.2.9) and step 1 are proven.

Case 1 – Step 2. Define the $*$-homomorphism $\zeta_\ell = \pi_\ell \circ \Phi : M \to B(\mathcal{H})$ and the $*$-antihomomorphism $\zeta_r = \pi_r \circ \Phi : M \to B(\mathcal{H})$. Define

$$S = \big[\zeta_\ell(M_0)\, \zeta_r(M_0)\, \pi\big(C_0(G)\big)\, W_v \mid v \in \mathcal{N}_{pMp}(A)\big]\ . \tag{4.2.13}$$

Finally, define the isometry $Z \in B(\mathcal{H}, \mathcal{H} \otimes L^2(G))$ given by $Z = (\pi \otimes \mathrm{id})(Z_0)$. We claim that $S \subseteq B(\mathcal{H})$ is a C*-algebra, that

$$Z\zeta_\ell(x) - (\zeta_\ell \otimes \mathrm{id})\big(\Phi(x)\big)Z \in S \otimes_{\min} L^2(G) \tag{4.2.14}$$

and that

$$Z\zeta_r(x) - (\zeta_r(x) \otimes 1)Z \in S \otimes_{\min} L^2(G) \tag{4.2.15}$$

for all $x \in M_0$.

Since $\zeta_\ell : M_0 \to B(\mathcal{H})$ and $\pi : C_0(G) \to B(\mathcal{H})$ are covariant w.r.t. the continuous coaction $\Phi : M_0 \to M(M_0 \otimes_{\min} C_r^*(G))$, they induce a nondegenerate representation of the full crossed product. Since G is coamenable, the canonical homomorphisms of the full crossed product onto the reduced crossed product is an isomorphism. The reduced crossed product is given by the C*-algebra S_ℓ defined in (4.2.7). So, we find a nondegenerate $*$-homomorphism $\theta_\ell : S_\ell \to B(\mathcal{H})$ such that

$$\theta_\ell\big(\Phi(x)(1 \otimes F)\big) = \zeta_\ell(x)\pi(F),$$

for all $x \in M_0$ and $F \in C_0(G)$.

Associated with the coaction $\Phi : M \to M \overline{\otimes} L(G)$, we have the canonical coaction $\Phi^{\text{op}} : M^{\text{op}} \to M^{\text{op}} \overline{\otimes} R(G)$ defined as follows. Denote by $\gamma : M \to M^{\text{op}} : x \mapsto x^{\text{op}}$ the canonical $*$-anti-isomorphism. Also define the $*$-anti-isomorphism $\eta : L(G) \to R(G) : x \mapsto Jx^*J$, where J is the modular conjugation. Note that $\eta(\lambda_g) = \rho_{g^{-1}}$ for all $g \in G$. Then, Φ^{op} is defined by $\Phi^{\text{op}}(x^{\text{op}}) = (\gamma \otimes \eta) \circ \Phi(x)$. The corresponding crossed product C*-algebra is

$$S_r = [\Phi^{\text{op}}(M_0^{\text{op}})(1 \otimes C_0(G))] \subseteq M^{\text{op}} \overline{\otimes} B(L^2(G)) \; .$$

Since also ζ_r and π are covariant, we similarly find a nondegenerate $*$-homomorphism $\theta_r : S_r \to B(\mathcal{H})$ satisfying

$$\theta_r(\Phi^{\text{op}}(x^{\text{op}})(1 \otimes F)) = \zeta_r(x)\pi(F) \; ,$$

for all $x \in M_0, F \in C_0(G)$.

So $\theta_\ell(S_\ell) = [\zeta_\ell(M_0)\pi(C_0(G))]$ and $\theta_r(S_r) = [\zeta_r(M_0)\pi(C_0(G))]$ and these are C*-algebras. Moreover, the unitaries W_v for $v \in \mathcal{N}_{pMp}(A)$, commute with $\zeta_\ell(M)$, $\zeta_r(M)$ and $\pi(C_0(G))$. So, the space S defined in (4.2.13) is a C*-algebra and

$$S = [\theta_\ell(S_\ell)\, \theta_r(S_r)\, W_v \mid v \in \mathcal{N}_{pMp}(A)] \; .$$

Also, $\theta_\ell(S_\ell) \subseteq S$ and $\theta_r(S_r) \subseteq S$.

Applying $\theta_\ell \otimes \text{id}$ to (4.2.9), we find (4.2.14). In the same way as we proved (4.2.9), one proves that

$$(1 \otimes Z_0)\Phi^{\text{op}}(x^{\text{op}}) - (\Phi^{\text{op}}(x^{\text{op}}) \otimes 1)(1 \otimes Z_0) \in S_r \otimes_{\min} L^2(G) \qquad (4.2.16)$$

for all $x \in M_0$. Applying $\theta_r \otimes \text{id}$ to (4.2.16), also (4.2.15) follows and step 2 is proven.

Case 1 – Notations. Write $\mathcal{G} = \mathcal{N}_{pMp}(A)$ and consider the $*$-algebras $\mathbb{C}\mathcal{G}$ and $D = M \otimes_{\text{alg}} M^{\text{op}} \otimes_{\text{alg}} \mathbb{C}\mathcal{G}$. Define the $*$-homomorphisms

$$\Theta : D \to B(\mathcal{H}) : x \otimes y^{\text{op}} \otimes v \mapsto \zeta_\ell(x)\, \zeta_r(y)\, W_v \;\; ,$$

$$\Theta_1 : D \to B(\mathcal{H} \otimes L^2(G)) : x \otimes y^{\text{op}} \otimes v \mapsto (\zeta_\ell \otimes \text{id})\big(\Phi(x)\big)\, (\zeta_r(y)\, W_v \otimes 1) \;\; .$$

Choose a positive functional Ω on $B(\mathcal{H})$ as a weak* limit point of the net of vector functionals $T \mapsto \langle T\xi_n, \xi_n\rangle$.

Case 1 – Step 3. Writing $C = \|\Omega\| \, \Lambda(G)^2$, we claim that

$$|\Omega(\Theta(x))| \leqslant C \, \|\Theta_1(x)\| \quad \text{for all } \; x \in D \, . \tag{4.2.17}$$

Since W_v commutes with $\pi(C_0(G))$ for all $v \in \mathcal{G}$, we have $ZW_v = (W_v \otimes 1)Z$ for all $v \in \mathcal{G}$. Denoting $D_0 = M_0 \otimes_{\text{alg}} M_0^{\text{op}} \otimes_{\text{alg}} \mathbb{C}\mathcal{G}$, equations (4.2.14) and (4.2.15) imply that for $x \otimes y^{\text{op}} \otimes v \in D_0$, we have

$$\begin{aligned} Z^*\Theta_1(x \otimes y^{\text{op}} \otimes v)Z &= Z^*(\zeta_\ell \otimes \text{id})\big(\Phi(x)\big)(\zeta_r(y)W_v \otimes 1)Z \\ &= Z^*(\zeta_\ell \otimes \text{id})\big(\Phi(x)\big)Z\zeta_r(y)W_v + Z^*(\zeta_\ell \otimes \text{id})\big(\Phi(x)\big)T \\ &= Z^*Z\zeta_\ell(x)\zeta_r(y)W_v + Z^*(\zeta_\ell \otimes \text{id})\big(\Phi(x)\big)T + Z^*T', \\ &= Z^*Z\zeta_\ell(x)\zeta_r(y)W_v + \zeta_\ell(x)Z^*T + (T'')^*T + Z^*T' \end{aligned}$$

where the error terms T, T' and T'' belong to $S \otimes_{\min} L^2(G)$. Since Z is an isometry and $Z^* \in \pi(C_0) \otimes_{\min} L^2(G)^*$, it follows that

$$Z^*\Theta_1(x)Z - \Theta(x) \in S \tag{4.2.18}$$

for all $x \in D_0$. Since we are in case 1, we have that $\Omega\big(\pi(F)\big) = 0$ for all $F \in C_0(G)$. So, $\Omega(T) = 0$ for all $T \in S$. It then follows from (4.2.18) that

$$|\Omega(\Theta(x))| \leqslant \|\Omega\| \, \|\Theta_1(x)\| \quad \text{for all } \; x \in D_0 \, . \tag{4.2.19}$$

To conclude step 3, we now have to approximate as follows an arbitrary $x \in D$ by elements in D_0.

Take a net $(\eta_i)_i$ in $A(G)$ as in Lemma 4.2.5 and let $m_i : L(G) \to L(G)$ and $\varphi_i : M \to M$ be the associated normal completely bounded maps. Note that the image of Θ_1 lies in $B(\mathcal{H}) \overline{\otimes} L(G)$ and that $(\text{id} \otimes m_i) \circ \Theta_1 = \Theta_1 \circ (\varphi_i \otimes \text{id} \otimes \text{id})$. It follows that

$$\|\Theta_1\big((\varphi_i \otimes \text{id} \otimes \text{id})(x)\big)\| \leqslant \Lambda_G \, \|\Theta_1(x)\|$$

for all $x \in D$ sand all i. Denoting by $\rho : L(G) \to L(G)$ the period 2 anti-automorphism given by $\rho(\lambda_g) = \lambda_{g^{-1}}$, the representation Θ_1 is unitarily conjugate to the representation

$$\Theta_2 : D \to B(\mathcal{H} \otimes L^2(G)) : x \otimes y^{\text{op}} \otimes v \mapsto (\zeta_\ell(x) \otimes 1) \, (\zeta_r \otimes \rho)\big(\Phi(y)\big) \, (W_v \otimes 1).$$

So, writing $\varphi_j^{\text{op}}(y^{\text{op}}) = (\varphi_j(y))^{\text{op}}$, we also find that

$$\|\Theta_1\big((\text{id} \otimes \varphi_j^{\text{op}} \otimes \text{id})(x)\big)\| \leqslant \Lambda_G \, \|\Theta_1(x)\|$$

for all $x \in D$ and all j. Altogether, we have proved that

$$\|\Theta_1\big((\varphi_i \otimes \varphi_j^{\mathrm{op}} \otimes \mathrm{id})(x)\big)\| \leqslant \Lambda_G^2 \, \|\Theta_1(x)\|$$

for all $x \in D$ and all i, j. For every $T \in B(\mathcal{H})$, write $\|T\|_\Omega = \sqrt{\Omega(T^*T)}$. Since

$$\|\zeta_\ell\big(\varphi_i(x) - \zeta_\ell(x)\big)\|_\Omega^2 = \mathrm{Tr}(p(\varphi_i(x) - x)^*(\varphi_i(x) - x)p)$$

for every $x \in M$, it follows from the Cauchy-Schwarz inequality that for every $x \in D$ and every i,

$$\Omega\big(\Theta((\mathrm{id} \otimes \varphi_j^{\mathrm{op}} \otimes \mathrm{id})(x))\big) = \lim_i \Omega\big(\Theta((\varphi_i \otimes \varphi_j^{\mathrm{op}} \otimes \mathrm{id})(x))\big) \; .$$

Similarly, we have

$$\Omega(\Theta(x)) = \lim_j \Omega\big(\Theta((\mathrm{id} \otimes \varphi_j^{\mathrm{op}} \otimes \mathrm{id})(x))\big)$$

for all $x \in D$. Since $(\varphi_i \otimes \varphi_j^{\mathrm{op}} \otimes \mathrm{id})(x) \in D_0$ for all i, j, it follows from (4.2.19) that

$$|\Omega((\varphi_i \otimes \varphi_j^{\mathrm{op}} \otimes \mathrm{id})(x))| \leqslant \|\Omega\| \, \|\Theta_1((\varphi_i \otimes \varphi_j^{\mathrm{op}} \otimes \mathrm{id})(x))\| \leqslant C \, \|\Theta_1(x)\|$$

for all i, j. Taking first the limit over i and then over j, we find that (4.2.17) holds and step 3 is proven.

Case 1 – End of the proof. Because of (4.2.17), we can define a continuous functional Ω_1 on the C*-algebra $[\Theta_1(D)]$ satisfying $\Omega_1\big(\Theta_1(x)\big) = \Omega\big(\Theta(x)\big)$ for all $x \in D$. Since

$$\Omega_1\big(\Theta_1(x)^*\Theta_1(x)\big) = \Omega\big(\Theta(x^*x)\big) \geqslant 0$$

for all $x \in D$, it follows by density that Ω_1 is positive.

Extend Ω_1 to a functional on $B(\mathcal{H} \otimes L^2(G))$ without increasing its norm. So Ω_1 remains positive. Write $q_1 = (\zeta_\ell \otimes \mathrm{id})\big(\Phi(p)\big)\,(\zeta_r(p) \otimes 1)$. By construction, for every $v \in \mathcal{G}$, we have that $(\zeta_\ell \otimes \mathrm{id})(\Phi(v))(\zeta_r(v^*)W_v \otimes 1)$ is a unitary in $B\big(q_1(\mathcal{H} \otimes L^2(G))\big)$ with

$$\begin{aligned}
\Omega_1\big((\zeta_\ell \otimes \mathrm{id})(\Phi(v))(\zeta_r(v^*)W_v \otimes 1)\big) &= \Omega_1\big(\Theta_1(v \otimes (v^*)^{\mathrm{op}} \otimes v)\big) \\
&= \Omega\big(\Theta(v \otimes (v^*)^{\mathrm{op}} \otimes v)\big) \\
&= \mathrm{Tr}(p) \\
&= \Omega_1(q_1) \; .
\end{aligned}$$

Also, $\Omega_1(1 - q_1) = 0$ and $\Omega_1\big((\zeta_\ell \otimes \mathrm{id})(\Phi(x))\big) = \mathrm{Tr}(pxp)$ for all $x \in M$.

Define the positive functional Ω_2 on $q\big(M \mathbin{\overline{\otimes}} B(L^2(G))\big)q$ given by $\Omega_2(T) = \Omega_1\big((\zeta_\ell \otimes \mathrm{id})(T)\big)$. Then, $\Omega_2\big(\Phi(x)\big) = \mathrm{Tr}(x)$ for all $x \in pMp$ and Ω_2 is $\Phi(\mathcal{G})$-central. Writing $P = \mathcal{N}_{pMp}(A)''$, the Cauchy-Schwarz inequality implies that Ω_2 is $\Phi(P)$-central. So we have proved that P is Φ-amenable.

Proof in Case 2. After passing to a subnet, we may assume that there is an $F \in C_0(G)$ such that the net $(\|\pi(F)\xi_i\|)_i$ is convergent to a strictly positive number. Let Ω be a weak* limit point of the net of functionals $(\omega_i)_i$ defined by $\omega_i(T) = \langle T\xi_i, \xi_i\rangle$. As before, let $\zeta_\ell = \pi_\ell \circ \Phi$. Define the C*-algebra $S_1 := \theta_\ell(S_\ell) = [\zeta_\ell(M_0)\pi(C_0(G))]$. Denote by Ω_1 the restriction of Ω to S_1''. By construction, $\Omega_1(\zeta_\ell(x)) = \mathrm{Tr}(pxp)$ for all $x \in M$. Moreover Ω_1 is $\zeta_\ell(A)$-central, since

$$
\begin{aligned}
\Omega_1\big(\zeta_\ell(v)x\zeta_\ell(v^*)\big) &= \lim_i \omega_i\big(\zeta_\ell(v)x\zeta_\ell(v^*)\big) \\
&= \lim_i \big\langle x\pi_\ell\big(\Phi(v^*)\big)\xi_i, \pi_\ell\big(\Phi(v^*)\big)\xi_i\big\rangle \\
&= \lim_i \big\langle x\pi_\ell\big(\rho(v)\big)\xi_i, \pi_\ell\big(\rho(v)^*\big)\xi_i\big\rangle \\
&= \lim_i \langle x\xi_i, \xi_i\rangle
\end{aligned}
$$

and hence

$$
\omega_i\big(\zeta_\ell(v)x\zeta_\ell(v^*)\big) = \big\langle x\pi_\ell\big(\Phi(v^*)\big)\xi_i, \pi_\ell\big(\Phi(v^*)\big)\xi_i\big\rangle
$$

for all $v \in \mathcal{U}(A)$ and $x \in S_1''$.

Define $\delta = \|\Omega_1|_{S_1}\|$ and put $\varepsilon = \delta(4\Lambda(G)^3 + 2\Lambda(G)^2 + 2)^{-1}$. Since the elements $\pi(f)$, with $f \in C_c(G)$ and $0 \leqslant f \leqslant 1$, form an approximate identity for S_1, we can fix $f \in C_c(G)$ with $0 \leqslant f \leqslant 1$ and

$$
\Omega_1(\pi(f)) \geqslant \delta - \varepsilon \qquad \text{and} \qquad \big|\Omega_1(T) - \Omega_1\big(T\pi(f)\big)\big| < \varepsilon\,\|T\|
$$

for all $T \in S_1$. As above, take a net of completely bounded maps $\varphi_i : M \to M$ such that $\|\varphi_i\|_{\mathrm{cb}} \leqslant \Lambda(G)$ and $\varphi_i(M) \subseteq M_0$ for all i and $\varphi_i(x) \to x$ strongly for all $x \in M$. Because $\Omega_1(\zeta_\ell(x)) = \mathrm{Tr}(pxp)$ for all $x \in M$,

$$
\Omega_1\big(\zeta_\ell(x)T\zeta_\ell(y)\big) = \lim_i \Omega_1\big(\zeta_\ell(\varphi_i(x))T\zeta_\ell(\varphi_i(y))\big) \tag{4.2.20}
$$

for all $x, y \in M$ and $T \in S_1''$.

We then find, for all $u \in \mathcal{U}(A)$,

$$
\delta \leqslant \Omega_1(\pi(f)) + \varepsilon
$$

$$= \Omega_1\big(\zeta_\ell(u^*)\pi(F)\zeta_\ell(u)\big) + \varepsilon$$
$$= \lim_i \operatorname{Re} \Omega_1\big(\zeta_\ell(\varphi_i(u^*))\pi(f)\zeta_\ell(\varphi_i(u))\big) + \varepsilon \; .$$

Since $\zeta_\ell\big(\varphi_m(u^*)\big)\pi(f)\zeta_\ell\big(\varphi_n(u)\big)$ belongs to S_1 and has norm at most $\Lambda(G)^2$, we get that

$$\delta \leqslant \limsup_i \operatorname{Re} \Omega_1\big(\zeta_\ell(\varphi_i(u^*))\pi(f)\zeta_\ell(\varphi_i(u))\pi(f)\big) + \varepsilon(\Lambda(G)^2 + 1) \; . \tag{4.2.21}$$

We claim that there exists an $\omega_0 \in A(G)$ such that the corresponding completely bounded map $\varphi_0 = (\mathrm{id} \otimes \omega_0) \circ \Phi : M \to M$: satisfies $\|\varphi_0\|_{\mathrm{cb}} \leqslant 2\Lambda(G)$ and

$$\pi(f)\zeta_\ell(x)\pi(f) = \pi(f)\zeta_\ell(\varphi_0(x))\pi(f) \quad \text{for all } \; x \in M_0 \; . \tag{4.2.22}$$

Using $\theta_\ell : S_\ell \to S_1$, it suffices to construct $\omega_0 \in A(G)$ such that $\|\varphi_0\|_{\mathrm{cb}} \leqslant 2\Lambda(G)$ and

$$(1 \otimes f)\Phi(x)(1 \otimes f) = (1 \otimes f)\Phi\big(\varphi_0(x)\big)(1 \otimes f) \quad \text{for all } \; x \in M_0 \; . \tag{4.2.23}$$

Denote by $K \subseteq G$ the (compact) support of f. By [CH89, Proposition 1.1], we can choose $\omega_0 \in A(G)$ such that $\omega_0(g) = 1$ for all $g \in KK^{-1}$ and such that the map $m_0 = (\mathrm{id} \otimes \omega_0) \circ \Delta$ satisfies $\|m_0\|_{\mathrm{cb}} \leqslant 2\Lambda(G)$. Viewing f as a multiplication operator on $L^2(G)$, we have that $f\lambda_g f = 0$ for all $g \in G \setminus KK^{-1}$. It follows that $fxf = fm_0(x)f$ for all $x \in L(G)$. We then also have

$$(1 \otimes f)\Phi(x)(1 \otimes f) = (1 \otimes f)\,(\mathrm{id} \otimes m_0)(\Phi(x))\,(1 \otimes f) = (1 \otimes f)\Phi(\varphi_0(x))(1 \otimes f) \; .$$

So (4.2.23) holds and (4.2.22) is proved.

Combining (4.2.22) and (4.2.21) and using that $\zeta_\ell\big(\varphi_i(u^*)\big)\pi(f)\zeta_\ell\big(\varphi_0(\varphi_i(u))\big)$ is an element of S_1 with norm at most $2\Lambda(G)^3$, we get that

$$\delta \leqslant \limsup_i \operatorname{Re} \Omega_1\big(\zeta_\ell(\varphi_i(u^*))\pi(f)\zeta_\ell(\varphi_0(\varphi_i(u)))\pi(f)\big) + \varepsilon(\Lambda(G)^2 + 1)$$
$$\leqslant \limsup_i \operatorname{Re} \Omega_1\big(\zeta_\ell(\varphi_i(u^*))\pi(f)\zeta_\ell(\varphi_0(\varphi_i(u)))\big) + \delta/2 \; .$$

As in (4.2.20) and using the $\zeta_\ell(A)$-centrality of Ω_1, we conclude that

$$\delta/2 \leqslant \operatorname{Re} \Omega_1\big(\zeta_\ell(u^*)\pi(f)\zeta_\ell(\varphi_0(u))\big) = \operatorname{Re} \Omega_1\big(\pi(f)\zeta_\ell(\varphi_0(u)u^*)\big)$$

for all $u \in \mathcal{U}(A)$. For every $T \in S_1''$ and $x \in M$, we have

$$|\Omega_1\big(T\zeta_\ell(x)\big)|^2 \leqslant \Omega_1(TT^*) \; \mathrm{Tr}(px^*xp) \; .$$

So we find a unique $\eta \in L^2(Mp)$ such that

$$\Omega_1\big(\pi(f)\zeta_\ell(x)\big) = \langle \widehat{xp}, \eta \rangle \quad \text{for all } \; x \in M \; .$$

It then follows that

$$\delta/2 \leqslant \operatorname{Re}\langle \varphi_0(u)u^*, \eta \rangle \quad \text{for all } \; u \in \mathcal{U}(A) \; .$$

Writing $\omega_0(g) = \langle \lambda_g \xi_1, \xi_2 \rangle$ with $\xi_1, \xi_2 \in L^2(G)$, this means that

$$\delta/2 \leqslant \operatorname{Re}\langle \Phi(u)(\hat{p} \otimes \xi_1)u^*, \eta \otimes \xi_2 \rangle \quad \text{for all } \; u \in \mathcal{U}(A) \; .$$

Defining $\xi_3 \in \Phi(p)(L^2(Mp) \otimes L^2(G))$ as the element of minimal norm in the closed convex hull of $\{\Phi(u)(p \otimes \xi_1)u^* \mid u \in \mathcal{U}(A)\}$, it follows that ξ_3 satisfies $\Phi(u)\xi_3 = \xi_3 u$ for all $u \in A$ and that $\operatorname{Re}\langle \xi_3, \eta \otimes \xi_2 \rangle \geqslant \delta/2$. So, $\xi_3 \neq 0$ and we have proven that A can be Φ-embedded. □

4.3 Uniqueness of Cartan subalgebras

We are now ready to prove our uniqueness theorem for Cartan subalgebras in the group measure space construction of locally compact groups that are weakly amenable and in class $\mathcal{S}$ (Theorem F from the introduction). The actual result that we prove here is more general than the one stated in the introduction. To formulate this more general result, recall that a nonsingular action $G \curvearrowright (X, \mu)$ of a locally compact group G on a standard probability space is called *amenable in the sense of Zimmer* if there exists a G-equivariant conditional expectation $E : L^\infty(X \times G) \to L^\infty(X)$ w.r.t. the action $G \curvearrowright X \times G$ given by $g \cdot (x, h) = (gx, gh)$.

Theorem 4.3.1. *Let $G = G_1 \times \cdots \times G_n$ be a direct product of nonamenable, weakly amenable, locally compact groups in class $\mathcal{S}$. Let $G \curvearrowright (X, \mu)$ be an essentially free nonsingular action. Denote by G_i° the direct product of all G_j, $j \neq i$.*

If for every $i \in \{1, \dots, n\}$ and every nonnull G-invariant Borel set $X_0 \subseteq X$, the action $G_i \curvearrowright L^\infty(X_0)^{G_i^\circ}$ is nonamenable in the sense of Zimmer, then $L^\infty(X) \rtimes G$ has a unique Cartan subalgebra up to unitary conjugacy.

In particular, $L^\infty(X) \rtimes G$ has a unique Cartan subalgebra up to unitary conjugacy when the groups G_i are nonamenable and the action $G \curvearrowright (X, \mu)$ is either probability measure preserving or irreducible.

To prove Theorem 4.3.1, we will make use of the notion of a cross section from Section 2.5.5. Recall that for a cross section $X_1 \subseteq X$, the associated cross section equivalence relation $\mathcal{R} = \mathcal{R}(G \curvearrowright X) \cap (X_1 \times X_1)$ has countable orbits. We will first translate the dichotomy from Theorem 4.2.2 into a dichotomy on the normalizer of a subalgebra inside $L(\mathcal{R})$ whenever $\mathcal{R}$ is equipped with a cocycle $\omega : \mathcal{R} \to G$ into a weakly amenable group in class $\mathcal{S}$ (Theorem 4.3.5). We then apply this dichotomy result to the von Neumann algebra $L(\mathcal{R})$ associated to the cross section equivalence relation $\mathcal{R}$ of the (Maharam extension of) $G \curvearrowright (X, \mu)$ to obtain uniqueness of the Cartan subalgebra $L(\mathcal{R})$. By Corollary 2.5.42, we then deduce uniqueness of the Cartan subalgebra in $L^\infty(X) \rtimes G$.

ω-compactness

Given a countable, pmp equivalence relation $\mathcal{R}$ on (X, μ), we have the natural embedding $L(\mathcal{R}) \hookrightarrow L^2(\mathcal{R})$ given by $x \mapsto x1_\Delta$, where 1_Δ denotes the characteristic function of the diagonal $\Delta = \{(w, w)\}_{w \in X}$. The $\|.\|_2$ on $L(\mathcal{R})$ is defined as

$$\|x\|_2 = \tau(x^* x),$$

where τ denotes the natural trace associated to μ.

Inspired by [Vae13, Definition 2.2], we define the following.

Definition 4.3.2. Let $\mathcal{R}$ be a countable, pmp equivalence relation, G a locally compact group and $\omega : \mathcal{R} \to G$ a cocycle. We say that a subset $\mathcal{V} \subseteq M$ is *ω-compact* if for every $\varepsilon > 0$, there exists a compact subset $K \subseteq G$ such that

$$\|x - P_K^\omega(x)\|_2 \leqslant \varepsilon \, \|x\|$$

for all $x \in \mathcal{V}$. Here P_K^ω denotes the orthogonal projection of $L^2(\mathcal{R})$ onto $L^2(\omega^{-1}(K))$.

We have the following result.

Lemma 4.3.3. *Let $\mathcal{R}$ be a countable, pmp equivalence relation and $\omega : \mathcal{R} \to G$ a cocycle. Denote $M = L(\mathcal{R})$. If $\mathcal{V} \subseteq M$ is a $\|.\|$-bounded set that is ω-compact, then also $x\mathcal{V}y$ is ω-compact for all $x, y \in M$.*

Proof. Without loss of generality, we can assume that $\mathcal{V}$ is in the unit ball of M. By symmetry, it suffices to prove the result for $y = 1$. Since $\text{span}\{u_\varphi\}_{\varphi \in [[\mathcal{R}]]}$ is $\|.\|_2$-dense in M, it suffices to prove the result for $x = u_\varphi$ with $\varphi : A \to B$ in $[[\mathcal{R}]]$. Take an arbitrary $\varepsilon > 0$ and fix $K \subseteq G$ such that

$$\|z - P_K^\omega(z)\| \leqslant \frac{\varepsilon}{2} \, \|z\|$$

for any $z \in \mathcal{V}$. By Lemma 2.5.12, we can take a compact subset $L \subseteq G$ and a Borel subset $E \subseteq X$ such that $\mu(E) \geqslant 1 - \varepsilon/2$ and $\omega\big(\varphi^{-1}(x), x\big) \in L$ for all $x \in E \cap A$. Denote by $q \in M$ the characteristic function of E. For every $z \in \mathcal{V}$, we have

$$\begin{aligned}\|u_\varphi z - P_{L^{-1}K}(u_\varphi z)\|_2 &\leqslant \big\|(1-q)u_\varphi z - P^\omega_{L^{-1}K}\big((1-q)u_\varphi z\big)\big\|_2 \\ &\quad + \|qu_\varphi z - P^\omega_{L^{-1}K}(qu_\varphi z)\|_2 \\ &\leqslant \|1-q\|_2 \, \|z\| + \|z - P^\omega_{\mathcal{K}}(z)\|_2 \\ &\leqslant \varepsilon,\end{aligned}$$

where we used that

$$\begin{aligned}\|qu_\varphi z - P^\omega_{L^{-1}K}(qu_\varphi z)\|_2^2 &= \int_{X_1} \sum_{y \sim x} q(x) p^\omega_{L^{-1}K}(x,y) |z\big(\varphi^{-1}(x), y\big)|^2 \, \mathrm{d}\mu_1(x) \\ &= \int_{X_1} \sum_{y \sim x} q\big(\varphi(x)\big) p^\omega_{G \setminus L^{-1}K}\big(\varphi(x), y\big) |z(x,y)|^2 \, \mathrm{d}\mu_1(x) \\ &\leqslant \int_{X_1} \sum_{y \sim x} p^\omega_{G \setminus K}(x,y) |z(x,y)|^2 \, \mathrm{d}\mu_1(x) \\ &= \|z - P^\omega_K(z)\|_2^2 .\end{aligned}$$

Here, p^ω_A denote the characteristic function of $\omega^{-1}(A)$ for $A \subseteq G$. We conclude that the result holds. ☐

The following result is an adaptation of [Vae13, Proposition 2.6] to this context. The proof is almost exactly the same, but for completeness we provide the full proof.

Proposition 4.3.4. *Let $\mathcal{R}$ be a countable, pmp equivalence relation and $\omega : \mathcal{R} \to G$ a cocycle. Denote $M = L(\mathcal{R})$. Let $B \subseteq pMp$ be a von Neumann subalgebra. The set of projections*

$$\mathcal{P} = \{q \in B' \cap pMp \mid q \text{ is a projection and } Bq \text{ is } \omega\text{-compact}\}$$

has a unique maximum $q \in \mathcal{P}$. Moreover, $q \in \mathcal{N}_{pMp}(B)' \cap pMp$.

Proof. Using Zorn's lemma, one finds a maximal orthogonal family $\{q_i\}_{i \in I}$ of projections in $\mathcal{P}$. Set $q = \sum_{i \in I} q_i$. We have $q \in \mathcal{P}$. Indeed, take $\varepsilon > 0$. Take a finite subset $I_0 \subseteq I$ such that $q_0 = \sum_{i \in I_0} q_i$ satisfies $\|q - q_0\|_2^2 \leqslant$

$\sum_{i \in I \setminus I_0} \|q_i\|_2^2 < \varepsilon^2/4$. Since all we assumed $q_i \in \mathcal{P}$ for all $i \in I_0$, we find a compact sets $K_i \subseteq G$ such that

$$\left\| b - P^{\omega}_{K_i}(b) \right\|_2 \leqslant \frac{\varepsilon}{2|I_0|} \|b\|$$

for every $b \in Bq_i$ and every $i \in I_0$. Set $K = \bigcup_{i \in I_0} K_i$. Then, for every $b \in Bq$, we have

$$\begin{aligned} \|b - P^{\omega}_K(b)\|_2 &\leqslant \left\| b(q - q_0) - P^{\omega}_K\big(b(q - q_0)\big) \right\|_2 + \sum_{i \in I_0} \|bq_i - P^{\omega}_K(bq_i)\|_2 \\ &\leqslant \frac{\varepsilon}{2} \|b\| + \sum_{i \in I_0} \frac{\varepsilon}{2|I_0|} \|b\| \\ &= \varepsilon \|b\| \end{aligned}$$

We prove that $q \geqslant q'$ for any $q' \in \mathcal{P}$ and hence that q is the unique maximum of $\mathcal{P}$. Suppose that $q' \nleqslant q$. Then, $T = (p - q)q'(p - q)$ is a nonzero operator in $B' \cap pMp$. Take $S \in B' \cap pMp$ such that $q'' = TS$ is a nonzero spectral projection of T. Note that q'' is orthogonal to q. Moreover,

$$(Bq'')_1 = (B)_1 q'' = (p - q)(B)_1 q'(p - q)S \subseteq (p - q)(Bq')_1(p - q)S$$

So, by Lemma 4.3.3, we have that $(Bq'')_1$ is ω-compact and hence Bq'' is. This contradicts the maximality of the family $\{q_i\}_{i \in I}$.

Finally, we prove that $q \in \mathcal{N}_{pMp}(B)' \cap pMp$. Take $u \in \mathcal{N}_{pMp}(B)$. Again Lemma 4.3.3, we see that $u(Bq)_1 u^*$ ω-compact. But $u(Bq)_1 u^* = (Buqu^*)_1$ and hence by maximality of q, we have $uqu^* \leqslant q$ and hence $q = uqu^*$. It follows that q commutes with $\mathcal{N}_{pMp}(B)$. □

We will deduce from Theorem 4.2.2 the following result. Recall from Example 4.1.3 that given a countable equivalence relation $\mathcal{R}$ and a cocycle $\omega : \mathcal{R} \to G$, the von Neumann algebra $L(\mathcal{R})$ is equipped with a coaction $\Phi_\omega : L(\mathcal{R}) \to L(\mathcal{R}) \mathbin{\overline{\otimes}} L(G)$.

Theorem 4.3.5. *Let $\mathcal{R}$ be a countable pmp equivalence relation on the standard probability space (X_1, μ_1). Let G be a weakly amenable locally compact group in class $\mathcal{S}$ and $\omega : \mathcal{R} \to G$ a cocycle. Write $M = L(\mathcal{R})$ and assume that $A \subseteq M$ is a Φ_ω-amenable von Neumann subalgebra with normalizer $P = \mathcal{N}_M(A)''$. Denote by $p \in P' \cap M$ the unique maximal projection such that Ap is ω-compact. Then, $P(1 - p)$ is Φ_ω-amenable.*

Proof. By Theorem 4.2.2, we only have to prove the following statement: if $p \in A' \cap M$ is a nonzero projection such that Ap can be Φ_ω-embedded, then there exists a nonzero projection $q \in A' \cap M$ such that $q \leqslant p$ and Aq is ω-compact. So, take such a $p \in A' \cap M$. Since Ap can be Φ_ω-embedded, there exists a nonzero vector $\xi \in L^2(M) \otimes L^2(G)$ such that $\Phi_\omega(p)\xi = \xi = \xi(p \otimes 1)$ and such that $\Phi_\omega(a)\xi = \xi(a \otimes 1)$ for all $a \in Ap$.

Denote by q the smallest projection in M that satisfies $\xi = \xi(q \otimes 1)$. Note that q is uniquely defined. Since also $\xi(p \otimes 1) = \xi$ and $\xi(uqu^* \otimes 1) = \Phi(u)\xi(u^* \otimes 1) = \xi$ for any $u \in \mathcal{U}(A)$, it follows that $q \leqslant p$ and $q \in A' \cap M$. Note that q is nonzero, since $\xi \neq 0$. Consider the operator L_ξ defined as the closure of the operator $\hat{M} \to L^2(M) \otimes L^2(G) : \hat{x} \mapsto \xi x$. The polar decomposition of L_ξ yields a partial isometry $V : L^2(M) \to L^2(M) \otimes L^2(G)$ satisfying $V^*V = 1 - P_{\ker L_\xi} = q$. Moreover, this partial isometry commutes with the right M-actions on $L^2(M)$ and $L^2(M) \otimes L^2(G)$. Hence, identifying V with $V\hat{1}$, we can view $V \in M \mathbin{\overline{\otimes}} L^2(G)$. Since ξ is A-central, it also follows that $Va = \Phi_\omega(a)V$ for all $a \in Ap$.

Define $\mathcal{G} \subseteq [[\mathcal{R}]]$ consisting of all $\varphi \in [[\mathcal{R}]]$ for which the set

$$\{\omega(\varphi(x), x) \mid x \in \operatorname{dom}(\varphi)\}$$

has compact closure in G. For every $\varphi \in [\mathcal{R}]$ and every $\varepsilon > 0$, we can choose a Borel set $\mathcal{U} \subseteq X_1$ with $\mu_1(X_1 \setminus \mathcal{U}) < \varepsilon$ such that the restriction of φ to $\mathcal{U}$ belongs to $\mathcal{G}$. Therefore, the linear span of all fu_φ for $f \in L^\infty(X_1)$ and $\varphi \in \mathcal{G}$ defines a w.o. dense $*$-subalgebra M_0 of M. By construction, for every $x \in M_0$, there exists a compact subset $K \subseteq G$ such that $x = P_K^\omega(x)$.

Choose $\varepsilon > 0$. Consider the M-module $M \mathbin{\overline{\otimes}} L^2(G)$ from Section 4.1.2. Equip $M \mathbin{\overline{\otimes}} L^2(G)$ with the norm $\|\cdot\|_2$ given by the embedding $M \mathbin{\overline{\otimes}} L^2(G) \subseteq L^2(M) \otimes L^2(G)$, as well as the operator norm $\|\cdot\|_\infty$. By the Kaplansky density theorem for W*-modules (see Theorem 4.1.5), we can take $W \in M_0 \otimes_{\text{alg}} C_c(G) \subseteq M \mathbin{\overline{\otimes}} L^2(G)$ such that $\|W\|_\infty \leqslant 1$ and $\|V - W\|_2 < \varepsilon/3$ and $\|W^*W - q\|_2 < \varepsilon/3$. For every $a \in M$, we find that

$$\|Va - Wa\|_2 \leqslant \|V - W\|_2 \, \|a\| \leqslant \frac{\varepsilon}{3} \, \|a\|$$

and

$$\|\Phi_\omega(a)V - \Phi_\omega(a)W\|_2 \leqslant \|a\| \, \|V - W\|_2 \leqslant \frac{\varepsilon}{3} \, \|a\| \, .$$

Therefore, $\|\Phi_\omega(a)W - Wa\|_2 \leqslant \frac{2\varepsilon}{3} \, \|a\|$ for all $a \in Ap$. Since $\|W\|_\infty \leqslant 1$ and $\|W^*W - q\|_2 < \varepsilon/3$, we find that

$$\|W^*\Phi_\omega(a)W - aq\|_2 \leqslant \|W^*\Phi_\omega(a)W - aW^*W\|_2 + \|aW^*W - aq\|_2 \leqslant \varepsilon \, \|a\| \tag{4.3.1}$$

for all $a \in Ap$.

When $\xi_i \in C_c(G)$ have (compact) supports $K_i \subseteq G$, then we claim that $(1 \otimes \xi_2^*)\Phi_\omega(x)(1 \otimes \xi_1)$ belongs to $L^2(\omega^{-1}(K_2 K_1^{-1}))$ for all $x \in M$. Indeed, it suffices to prove that claim for $x = f u_\varphi$ with $f \in L^\infty(X)$ and $\varphi \in [\mathcal{R}]$. In that case, (4.1.2) and (4.1.3) yield

$$\begin{aligned}\big((1 \otimes \xi_2^*)\Phi_\omega(x)(1 \otimes \xi_1)1_\Delta\big)(x,y) \\ &= \int_G \big(\Phi_\omega(f u_\varphi)(1_\Delta \otimes \xi_1)\big)(x,y,g)\overline{\xi_2(g)}\,\mathrm{d}g \\ &= \int_{K_2} f(x) 1_\Delta\big(\varphi^{-1}(x),y\big)\xi_1\big(\omega(\varphi^{-1}(x),x)g\big)\overline{\xi_2(g)}\,\mathrm{d}g \\ &= f(x)1_\Delta\big(\varphi^{-1}(x),y\big)\int_{K_2}\xi_1\big(\omega(y,x)g\big)\overline{\xi_2(g)}\,\mathrm{d}g\end{aligned}$$

which is zero if $\omega(x,y) \in G \setminus K_2 K_1^{-1}$.

So because $W \in M_0 \otimes_{\text{alg}} C_c(G)$, we can take a compact subset $K \subseteq G$ such that $W^*\Phi_\omega(x)W$ belongs to the range of P_K^ω for every $x \in M$. It then follows from (4.3.1) that

$$\|P_K^\omega(aq) - aq\|_2 \leqslant \|W^*\Phi_\omega(a)W - aq\|_2 \leqslant \varepsilon\,\|a\|$$

for all $a \in Ap$. For every element $a \in Aq$, we can choose $a_1 \in Ap$ with $\|a_1\| = \|a\|$ and $a = a_1 q$. So we have proved that $\|P_K^\omega(a) - a\|_2 \leqslant \varepsilon\,\|a\|$ for all $a \in Aq$. Since $\varepsilon > 0$ was arbitrary, this means that Aq is ω-compact. □

In the formulation of Corollary 4.3.6 below, we make use of the following notion of an amenable pair of group actions, as introduced in [Ana82]. Let G be a locally compact group and let $G \curvearrowright (Y,\eta)$ and $G \curvearrowright (X,\mu)$ be nonsingular actions. Assume that $p : Y \to X$ is a G-equivariant Borel map such that the measures $p_*(\eta)$ and μ are equivalent. Following [Ana82, Définition 2.2], the pair (Y,X) is called amenable if there exists a G-equivariant conditional expectation $L^\infty(Y,\eta) \to L^\infty(X,\mu)$. In particular, the action $G \curvearrowright (X,\mu)$ is amenable in the sense of Zimmer if and only if the pair $(X \times G, X)$ with $g \cdot (x,h) = (gx, gh)$ is amenable.

Corollary 4.3.6. *Let G be a locally compact group and $G \curvearrowright (X,\mu)$ an essentially free, nonsingular action on the standard measure space (X,μ). Assume that the action scales the measure μ by the inverse of the modular function of G. Let (X_1,μ_1) be a partial cross section with $\mu_1(X_1) < \infty$ and denote by $\mathcal{R}$ the cross section equivalence relation on (X_1,μ_1), which is a countable equivalence relation with invariant probability measure $\mu_1(X_1)^{-1}\,\mu_1$ by Proposition 2.5.38 (b).*

Let H be a weakly amenable locally compact group in class $\mathcal{S}$ and $\pi : G \to H$ a continuous group homomorphism. Denote by $\omega : \mathcal{R} \to H$ the cocycle given by the composition of π and the canonical cocycle $\omega_0 : \mathcal{R} \to G$ determined by $\omega_0(x', x) \cdot x = x'$ for all $(x', x) \in \mathcal{R}$.

Let $A \subseteq L(\mathcal{R})$ be a Cartan subalgebra. If A is not ω-compact, then there exists a nonnull G-invariant Borel set $X_0 \subseteq X$ and a G-equivariant conditional expectation $L^\infty(X_0 \times H) \to L^\infty(X_0)$ w.r.t. the action $g \cdot (x, h) = (g \cdot x, \pi(g)h)$.

Proof. Write $M = L(\mathcal{R})$. Assume that the Cartan subalgebra $A \subseteq M$ is not ω-compact. Let $p \in \mathcal{Z}(M)$ be the unique maximal projection such that Ap is ω-compact. Then, by Theorem 4.3.5, we can take a have for $q = 1 - p \in \mathcal{Z}(M)$ that Mq is Φ_ω-amenable. Since $\mathcal{Z}(M) \subseteq L^\infty(X_1)$ we can take $X_2 \subseteq X_1$ such that $q = 1_{X_2}$, where $X_2 \subseteq X_1$ is an $\mathcal{R}$-invariant Borel set. Put $X_0 = G \cdot X_2$. Then X_0 is a nonnull G-invariant Borel set. We prove that there exists a G-equivariant conditional expectation $L^\infty(X_0 \times H) \to L^\infty(X_0)$.

Note that since $q \in L^\infty(X_1)$, we have $\Phi_\omega(q) = q \otimes 1$. The Φ_ω-amenability of Mq, together with Theorem 2.4.64 yields a conditional expectation $Mq \mathbin{\overline{\otimes}} B\big(L^2(H)\big) \to \Phi_\omega(Mq)$. Since (X_1, μ_1) is a partial cross section, we can choose a compact neighborhood K of e in G such that the action map $\theta : K \times X_1 \to X : (k, x) \mapsto k \cdot x$ is injective.

Write $N = L^\infty(X) \rtimes G \subseteq B\big(L^2(G) \otimes L^2(X)\big)$. Define the unitary $U_\pi \in \mathcal{U}(L^2(G) \otimes L^2(X) \otimes L^2(H))$ by

$$(U_\pi \xi)(g, x, h) = \xi(g, x, \pi(g)h)$$

and consider the coaction

$$\Phi_\pi : N \to N \otimes L(H) : \Phi_\pi(x) = U_\pi^*(x \otimes 1)U_\pi$$

A similar calculation as in Example 4.1.2 shows that

$$\Phi_\pi(f u_g) = f u_g \otimes u_{\pi(g)}$$

for all $f \in L^\infty(X)$ and $g \in G$ such that indeed Φ_π is a coaction. Define the projection $q_1 \in L^\infty(X)$ given by $q_1 = 1_{K \cdot X_1}$. In the proof of Proposition 2.5.41 (see Appendix A), an explicit isomorphism

$$\Psi : q_1 N q_1 \to B(L^2(K)) \mathbin{\overline{\otimes}} M \tag{4.3.2}$$

is constructed. Fixing Borel maps $\pi : X_0 \to X_1$ and $\gamma : X_0 \to G$ satisfying $x = \gamma(x)\pi(x)$ for a.e. $x \in X_0$, and $\gamma(gx) = g$ and $\pi(gx) = x$ for $x \in X$ and $g \in K$, this isomorphism is given by

$$\big(\Psi(q_1 u_g q_1)\xi\big)(k, x, y) = \xi\big(\gamma(g^{-1}kx), \pi(g^{-1}kx), y\big) q_1(g^{-1}kx)$$

and

$$\big(\Psi(aq_1)\xi\big)(k,x,y) = a(kx)\xi(k,x,y)$$

for $\xi \in L^2(K) \otimes L^2(M)$, $a \in L^\infty(X)$, $g \in G$, a.e. $(x,y) \in \mathcal{R}$ and a.e. $k \in K$. Denoting by $V \in L^\infty(K) \mathbin{\overline{\otimes}} L(H)$ the unitary given by $V(k) = \lambda_{\pi(k)}$ for a.e. $k \in K$, a straightforward calculation shows that

$$V_{13}(\mathrm{id} \otimes \Phi_\omega)\big(\Psi(x)\big)V_{13}^* = \Phi_\pi(x)$$

for $x \in q_1 N q_1$. Moreover, $\Psi(q_2) = 1 \otimes q$. We thus conclude that there exists a conditional expectation $E_2 : q_2 N q_2 \mathbin{\overline{\otimes}} B(L^2(H)) \to \Phi_\pi(q_2 N q_2)$.

Recall that $X_0 = G \cdot X_2$. Since $u_g q u_g^* = u_g 1_{X_2} u_g^* = 1_{g \cdot X_2}$, the projection $q_0 = 1_{X_0}$ is the central support of q_2 inside N. Hence, there also exists a conditional expectation $E : Nq_0 \mathbin{\overline{\otimes}} B(L^2(H)) \to \Phi_\pi(Nq_0)$. Indeed, we can take a family of orthogonal projections $\{p_i\}$ with $p_i \preceq q_2$ such that $\sum_i p_i = q_0$. Take partial isometries $u_i \in N$ such that $u_i^* u_i \leqslant q_2$ and $u_i u_i^* = p_i$. Then, we can define $E : Nq_0 \mathbin{\overline{\otimes}} B\big(L^2(H)\big) \to \Phi_\pi(Nq_0)$ by

$$E(x) = \sum_{i,j} \Phi_\omega(u_i) E_2\big(\Phi_\omega(u_i)^* x \Phi_\omega(u_j)\big)\Phi_\omega(u_j)^*$$

for $x \in Nq_0 \mathbin{\overline{\otimes}} B\big(L^2(H)\big)$.

We now restrict E to $L^\infty(X_0 \times H) \subseteq Nq_0 \mathbin{\overline{\otimes}} B(L^2(H))$. For all $f \in L^\infty(X_0 \times H)$ and $h \in L^\infty(X_0)$, we have

$$E(f)\Phi_\pi(h) = E\big(f\Phi_\pi(h)\big) = E\big(\Phi_\pi(h)f\big) = \Phi_\pi(h)E(f).$$

Since $L^\infty(X_0) \subseteq Nq_0$ is maximal abelian, it follows that $E(f) \in \Phi_\pi(L^\infty(X_0)) = L^\infty(X_0) \otimes 1$. Define the conditional expectation $E_0 : L^\infty(X_0 \times H) \to L^\infty(X_0)$ such that $E(F) = E_0(F) \otimes 1$. Since the action $G \curvearrowright L^\infty(X_0 \times H)$ is implemented by the unitary operators $\Phi_\pi(u_g q_0)$ for $g \in G$, it follows that E_0 is G-equivariant. This concludes the proof of the corollary. □

We prove the following relationship between the Cartan subalgebras of an arbitrary von Neumann algebra N and of its continuous core $\mathrm{c}(N) = N \rtimes_{\sigma^\varphi} \mathbb{R}$ from Definition 2.4.43.

Lemma 4.3.7. *Let N be a von Neumann algebra. Assume that the continuous core $\mathrm{c}(N) = N \rtimes_{\sigma^\varphi} \mathbb{R}$ has at most one Cartan subalgebra up to unitary conjugacy. Then the same holds for N itself.*

Proof. Let A and B be Cartan subalgebras of N. Denote by $E_A : N \to A$ and $E_B : N \to B$ the unique faithful normal conditional expectations. Let $z \in \mathcal{Z}(N)$

be a nonzero central projection. Note that $z \in A \cap B$ since both A and B are maximally abelian. The main part of the proof consists in showing that $Az \prec_{Nz} Bz$. Assuming that $Az \not\prec_{Nz} Bz$, we deduce a contradiction.

By Theorem 2.5.33, there exists a net of unitaries $u_n \in \mathcal{U}(Az)$ such that

$$E_B(x^* u_n y) \to 0 \tag{4.3.3}$$

in the strong* operator topology for all $x, y \in N$.

Choose faithful normal states φ on A and ψ on B. Still denote by φ and ψ the faithful normal states on N given by $\varphi \circ E_A$, resp. $\psi \circ E_B$. The continuous core of N can then be realized as $\mathrm{c}_\varphi(N) = N \rtimes_{\sigma^\varphi} \mathbb{R}$ and as $\mathrm{c}_\psi(N) = N \rtimes_{\sigma^\psi} \mathbb{R}$. Denote by $\Theta : \mathrm{c}_\varphi(N) \to \mathrm{c}_\psi(N)$ the canonical $*$-isomorphism given by Connes' Radon-Nikodym theorem.

Write $\widetilde{A} = \Theta(A \rtimes_{\sigma^\varphi} \mathbb{R})$ and $\widetilde{B} = B \rtimes_{\sigma^\psi} \mathbb{R}$. By Proposition 2.5.31, both $\widetilde{A}$ and $\widetilde{B}$ are Cartan subalgebras of $M = \mathrm{c}_\psi(N)$. Denote by Tr the canonical faithful normal semifinite trace on M and let $E_{\widetilde{B}} : M \to \widetilde{B}$ be the trace preserving conditional expectation. Since the restriction of $E_{\widetilde{B}}$ to N preserves the state ψ, we have $E_{\widetilde{B}}(x) = E_B(x)$ for all $x \in N$. We claim that

$$E_{\widetilde{B}}(x^* u_n y) \to 0 \tag{4.3.4}$$

in strong* operator topology for all $x, y \in M$. Since u_n is a net of unitaries in $\widetilde{A}z$, once this claim is proved, it follows that the Cartan subalgebras $\widetilde{A}$ and $\widetilde{B}$ cannot be unitarily conjugate, contradicting the assumptions of the theorem. Hence, the claim implies that $Az \prec_{Nz} Bz$.

To prove the claim, note that since $\widetilde{B}$ is abelian, it suffices to prove that

$$\lim_n \|E_{\widetilde{B}}(x^* u_n y)\|_{2,\mathrm{Tr}} = 0 \tag{4.3.5}$$

for all $x, y \in \mathfrak{n}$ where $\mathfrak{n} = \{x \in M \mid \mathrm{Tr}(x^* x) < \infty\}$ is the finite trace ideal of M. Indeed, suppose that (4.3.5) holds and take $x, y \in M$. Let $L^2(M)$ be the GNS-Hilbert space of M with respect to Tr and denote by $\eta : \mathfrak{n} \to L^2(M)$ the canonical embedding. Since $\{\eta(b'bz)\}_{b,b' \in \mathfrak{n} \cap \widetilde{B}, z \in \mathfrak{n}}$ is dense in $L^2(M)$ and the sequence $\left(E_{\widetilde{B}}(x^* u_n y)\right)_n$ is $\|.\|$-bounded, it suffices to prove that

$$\left\| E_{\widetilde{B}}(x^* u_n y)\eta(bb'z) \right\| \to 0 \qquad \text{and} \qquad \left\| E_{\widetilde{B}}(y^* u_n^* x)\eta(bb'z) \right\| \to 0$$

for all $b, b' \in \mathfrak{n} \cap \widetilde{B}$ and all $z \in \mathfrak{n}$. But,

$$\left\| E_{\widetilde{B}}(x^* u_n y)\eta(bb'z) \right\| = \left\| E_{\widetilde{B}}(x^* u_n y) bb'z \right\|_{2,\mathrm{Tr}} \leqslant \|z\|_\infty \left\| E_{\widetilde{B}}\big((xb^*)^* u_n y b'\big) \right\|_{2,\mathrm{Tr}}$$

and the right hand side converges to zero since $xb^* \in \mathfrak{n}$ and $yb' \in \mathfrak{n}$. By a similar calculation $\left\| E_{\widetilde{B}}(y^* u_n^* x)\eta(bb'z) \right\| \to 0$.

Approximating x and y in $\|\cdot\|_{2,\mathrm{Tr}}$, it suffices to prove (4.3.5) for all x, y of the form $x = x_1 x_0$ and $y = y_1 y_0$ with $x_1, y_1 \in N$ and $x_0, y_0 \in \widetilde{B}$ with $\mathrm{Tr}(x_0^* x_0) < \infty$ and $\mathrm{Tr}(y_0^* y_0) < \infty$. But then,

$$E_{\widetilde{B}}(x^* u_n y) = x_0^* \, E_{\widetilde{B}}(x_1^* u_n y_1) \, y_0 = x_0^* \, E_B(x_1^* u_n y_1) \, y_0 \ ,$$

so that (4.3.5) follows from (4.3.3).

Thus, (4.3.4) is proved. As we already explained, it follows that $Az \prec_{Nz} Bz$ for every nonzero central projection $z \in \mathcal{Z}(N)$.

Let $z_0 \in \mathcal{Z}(N)$ be the maximal central projection such that Az_0 and Bz_0 are unitarily conjugate inside Nz_0. Assume that $z_0 \neq 1$ and put $z = 1 - z_0$. By the above, $Az \prec_{Nz} Bz$. By Theorem 2.5.34, there exists a nonzero central projection $z_1 \in \mathcal{Z}(N)z$ such that Az_1 and Bz_1 are unitarily conjugate. This contradicts the maximality of z_0, so that $z_0 = 1$. □

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Take $G = G_1 \times \cdots \times G_n$ as in the formulation of the theorem. Let $G \curvearrowright (X, \mu)$ be an essentially free nonsingular action and assume that the hypotheses of the theorem hold. We have to prove that $N = L^\infty(X) \rtimes G$ has a unique Cartan subalgebra up to unitary conjugacy. By Lemma 4.3.7, it suffices to prove that the continuous core $\mathrm{c}(N)$ has a unique Cartan subalgebra up to unitary conjugacy.

The continuous core $\mathrm{c}(N)$ can be realized as a crossed product $\mathrm{c}(N) = L^\infty(\widetilde{X}) \rtimes G$ where $G \curvearrowright (\widetilde{X}, \widetilde{\mu})$ is the *Maharam extension* from [Mah64]. This is an action on the space $\widetilde{X} = X \times \mathbb{R}$ given by

$$g \cdot (x, t) = \big(gx, t + \log \delta_G(g) + \log D(g, x)\big)$$

for $g \in G$, $x \in X$ and $t \in \mathbb{R}$. Here, $\delta_G : G \to \mathbb{R}_0^+$ is the modular function of G and D is the Radon-Nikodym cocycle for $G \curvearrowright (X, \mu)$ as in [Zim84, Example 4.2.4] determined by

$$D(g, x) = \frac{\mathrm{d}(g^{-1} \cdot \mu)}{\mathrm{d}\mu}(x)$$

for a.e. $x \in X$ and a.e. $g \in G$. The measure $\widetilde{\mu}$ on $\widetilde{X}$ is given by $\widetilde{\mu} = \mu \otimes \eta$, where $\eta \in M(\mathbb{R})$ is given by $\mathrm{d}\eta(t) = \exp(-t) \, \mathrm{d}t$. Note that the action $G \curvearrowright \widetilde{X}$ scales the measure $\widetilde{\mu}$ with δ_G^{-1}.

Let (X_1, μ_1) be a cross section for $G \curvearrowright (\widetilde{X}, \widetilde{\mu})$. Denote by $\mathcal{R}_1$ the cross section equivalence relation on (X_1, μ_1). By Corollary 2.5.42, it suffices to prove that $L^\infty(X_1)$ is the unique Cartan subalgebra of $L(\mathcal{R}_1)$, up to unitary conjugacy. It suffices to prove that for every nonnull Borel set $X_2 \subseteq X_1$ with $\mu_1(X_2) < \infty$, the restricted equivalence relation $\mathcal{R} = (\mathcal{R}_1)|_{X_2}$ has the property that $L^\infty(X_2)$ is the unique Cartan subalgebra of $L(\mathcal{R})$ up to unitary conjugacy. Indeed, if this is the case, we take a family $\{Y_i\}_{i\in I}$ of subsets of X_1 with $\mu_1(Y_i) < +\infty$ such that the $\mathcal{R}_1$-saturations $Z_i = [Y_i]_{\mathcal{R}_1}$ are disjoint and $X_1 = \bigcup_i Z_i$. Writing $\mathcal{S}_i = (\mathcal{R}_1)|_{Z_i}$, we have, by Proposition 2.5.30, that each $L(\mathcal{S}_i)$ has unique Cartan subalgebra up to unitary conjugacy and hence also $L(\mathcal{R}_1)$ does.

So, take a nonnull Borel set $X_2 \subseteq X_1$ with $\mu_1(X_2) < \infty$. Denote by μ_2 the restriction of μ_1 to X_2. Then (X_2, μ_2) is a partial cross section for $G \curvearrowright (\widetilde{X}, \widetilde{\mu})$ and $\mu_2(X_2) < \infty$. Let $A \subseteq L(\mathcal{R})$ be another Cartan subalgebra.

Denote by $\omega : \mathcal{R} \to G$ the canonical cocycle determined by $\omega(x', x) \cdot x = x'$ for all $(x', x) \in \mathcal{R}$. We claim that A is ω-compact. Denote by $\pi_i : G \to G_i$ the quotient maps and put $\omega_i = \pi_i \circ \omega$. To prove the claim that A is ω-compact, it suffices to prove that A is ω_i-compact for every $i \in \{1, \dots, n\}$. Fix such an i and assume that A is not ω_i-compact.

By Corollary 4.3.6, we find a nonnull G-invariant Borel set $\widetilde{X}_0 \subseteq \widetilde{X}$ and a G-equivariant conditional expectation $E_0 : L^\infty(\widetilde{X}_0 \times G_i) \to L^\infty(\widetilde{X}_0)$ w.r.t. the action $g \cdot \big((x, t), g'\big) = \big(g \cdot (x, t), \pi_i(g)g'\big)$.

Denote by $(\beta_s)_{s\in\mathbb{R}}$ the action of $\mathbb{R}$ on $L^\infty(\widetilde{X})$ given by $s \cdot (g, t) = (g, t + s)$. Note that this action of $\mathbb{R}$ commutes with the above G-action. Write $p = 1_{\widetilde{X}_0}$ and denote by q the smallest $(\beta_s)_{s\in\mathbb{R}}$-invariant projection in $L^\infty(\widetilde{X})$ with $p \leqslant q$. Note that q is still G-invariant and hence $q = 1_{X_0 \times \mathbb{R}}$, where $X_0 \subseteq X$ is a G-invariant Borel set. Also note that $q = \bigvee_{g\in G} \beta_g(p)$. Take a maximal orthogonal family $\{q_i\}_{i\in I}$ of projections $q_i \in L^\infty(\widetilde{X})$ for which there exists a $s_i \in \mathbb{R}$ such that $q_i \leqslant \beta_{s_i}(p)$. Then, $q = \sum_i q_i$. Write $p_i = \beta_{-s_i}(q_i) \leqslant p$ and define for every finite subset $I_0 \subseteq I$ the G-equivariant positive linear map $E_{I_0} : L^\infty(X_0 \times \mathbb{R} \times G_i) \to L^\infty(X_0 \times \mathbb{R})$ by

$$E_{I_0}(f) = \sum_{i\in I_0} \beta_{s_i} \left(E_0\big((p_k \otimes 1)(\beta_{-s_i} \otimes 1)(f) \big) \right)$$

for any $f \in L^\infty(X_0 \times \mathbb{R} \times G_i)$. For every $f \in L^\infty(X_0 \times \mathbb{R})$ and finite $I_0 \subseteq I$, we have

$$E_{I_0}(f) = \sum_{i\in I_0} q_i f.$$

Hence, taking a point-weak* limit point of the net $(E_{I_0})_{I_0 \subseteq I}$, we obtain a G-equivariant conditional expectation $E : L^\infty(X_0 \times \mathbb{R} \times G_i) \to L^\infty(X_0 \times \mathbb{R})$.

Now, we construct a $G \times \mathbb{R}$-equivariant conditional expectation $E_1 : L^\infty(X_0 \times \mathbb{R} \times G_i) \to L^\infty(X_0 \times \mathbb{R})$ in the following way. Define $\tilde{E} : L^\infty(X_0 \times \mathbb{R} \times G_i) \to L^\infty(X_0 \times \mathbb{R} \times \mathbb{R})$ such that

$$\tilde{E}(f)(x,t,s) = \big(\beta_s \circ E \circ (\beta_{-s} \otimes 1)\big)(f)(x,t)$$

for a.e. $s, t \in \mathbb{R}$ and a.e. $x \in X_0$. Take a left $\mathbb{R}$-invariant mean $m : L^\infty(\mathbb{R}) \to \mathbb{C}$ and define $E_1 = (1 \otimes 1 \otimes m) \circ \tilde{E}$. One easily checks that E_1 is indeed a $G \times \mathbb{R}$-equivariant conditional expectation.

The restriction of E to $L^\infty(X_0 \times G_i) \subseteq L^\infty(X_0 \times \mathbb{R} \times G_i)$ then has its image in $L^\infty(X_0 \times \mathbb{R})^{\mathbb{R}} = L^\infty(X_0)$. So, we find a G-equivariant conditional expectation $L^\infty(X_0 \times G_i) \to L^\infty(X_0)$. Restricting to $L^\infty(X_0)^{G_i^\circ} \mathbin{\overline{\otimes}} L^\infty(G_i)$, we find a G_i-equivariant conditional expectation

$$L^\infty(X_0)^{G_i^\circ} \mathbin{\overline{\otimes}} L^\infty(G_i) \to L^\infty(X_0)^{G_i^\circ} .$$

This precisely means that the action $G_i \curvearrowright L^\infty(X_0)^{G_i^\circ}$ is amenable in the sense of Zimmer, contrary to our assumptions.

So the claim that A is ω-compact is proved. Take a compact subset $K \subseteq G$ such that $\|P_K^\omega(a)\|_2^2 \geqslant 1/2$ for all $a \in \mathcal{U}(A)$. Since K is compact and $\omega : \mathcal{R} \to G$ is the canonical cocycle, the subset $\omega^{-1}(K) \subseteq \mathcal{R}$ is bounded, meaning that $\omega^{-1}(K)$ is the disjoint union of the graphs of finitely many elements $\varphi_i \in [[\mathcal{R}]]$, $i = 1, \dots, n$, in the full pseudogroup of $\mathcal{R}$. But then, writing $B = L^\infty(X_2)$,

$$\|P_K^\omega(a)\|_2^2 = \sum_{i=1}^n \|E_B(a u_{\varphi_i}^*)\|_2^2$$

for all $a \in L(\mathcal{R})$. Since $\|P_K^\omega(a)\|_2^2 \geqslant 1/2$ for all $a \in \mathcal{U}(A)$, it follows that there exists no $(u_n)_n$ sequence in $\mathcal{U}(A)$ such that $\|E_B(u_n y)\|_2 \to 0$ for all $y \in L(\mathcal{R})$. This means that $A \prec_{L(\mathcal{R})} L^\infty(X_2)$, so that A and $L^\infty(X_2)$ are unitarily conjugate by Theorem 2.5.34.

Finally, we prove that the last two statements in Theorem 4.3.1 hold. Suppose first that $G \curvearrowright (X,\mu)$ is irreducible. Then, $L^\infty(X)^{G_i^\circ} = \mathbb{C}$. Since each G_i is nonamenable, also $G_i \curvearrowright L^\infty(X)^{G_i^\circ}$ is not amenable in the sense of Zimmer. Now, suppose that $G \curvearrowright (X,\mu)$ is pmp and that $G_i \curvearrowright L^\infty(X_0)^{G_i^\circ}$ is amenable in the sense of Zimmer for some i and some G-invariant Borel set $X_0 \subseteq X$. Let $E : L^\infty(X_0)^{G_i^\circ} \mathbin{\overline{\otimes}} L^\infty(G_i) \to L^\infty(X_0)^{G_i^\circ}$ be the corresponding G_i-equivariant conditional expectation. Denoting by φ the state on $L^\infty(X_0)$ given by integrating with respect to μ, the restriction of the composition $\varphi \circ E$ to $L^\infty(G_i)$ is a G_i-invariant mean on $L^\infty(G_i)$, thus contradicting the nonamenability of G_i. □

Combining Theorem F with [PV11, Proposition 7.1], we obtain the following first examples of II_∞ factors having a unique Cartan subalgebra up to unitary conjugacy, but not having a group measure space Cartan subalgebra, i.e. these factors do not decompose as group measure space von Neumann algebras by *discrete* groups.

Corollary 4.3.8. *Let $G = \mathrm{Sp}(n,1)$ with $n \geqslant 2$ and let $G \curvearrowright (X,\mu)$ be any weakly mixing Gaussian action. Put $M = L^\infty(X) \rtimes G$. Then, M is a II_∞ factor that has a unique Cartan subalgebra up to unitary conjugacy, but that has no group measure space Cartan subalgebra. In particular, its finite corners pMp are II_1 factors with unique Cartan subalgebra, but without group measure space Cartan subalgebra.*

Proof. Take a cross section $X_1 \subseteq X$ and denote by $\mathcal{R}$ the associated cross section equivalence relation. By Corollary 2.5.42, we have that

$$pMp \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big),$$

where $\mathcal{U} \subseteq X$ is a neighborhood of identity such that the action map $\mathcal{U} \times X_1 \to X$ is injective and $p = 1_{\mathcal{U}\cdot X_1}$. Since M is an infinite factor, we have

$$M \cong pMp \mathbin{\overline{\otimes}} B\big(\ell^2(\mathbb{N})\big) \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathbb{N})\big).$$

By Lemma 2.5.29 every finite corner of M is isomorphic to qMq for some finite projection $q \in L^\infty(X_1) \mathbin{\overline{\otimes}} \ell^2(\mathbb{N})$.

Fix such a $q \in L^\infty(X_1) \mathbin{\overline{\otimes}} \ell^2(\mathbb{N})$. We have $qMq \cong L(\mathcal{R}^t)$ for some $t > 0$, where $\mathcal{R}^t$ denotes the amplification of $\mathcal{R}$. By Theorem F, M has a unique Cartan subalgebra and by Proposition 2.5.30 so does the corner qMq. Hence, if $qMq \cong L^\infty(X) \rtimes \Gamma$ for some free, ergodic, pmp action $\Gamma \curvearrowright (X,\mu)$, then $\mathcal{R}^t \cong \mathcal{R}(\Gamma \curvearrowright X)$. However, by [PV11, Proposition 7.1], this is impossible. ☐

4.4 Cocycle rigidity, orbit equivalence rigidity and W*-strong rigidity

Given an irreducible pmp action of $G = G_1 \times G_2$ on a standard probability space (X,μ), Monod and Shalom proved in [MS04, Theorem 1.2] a cocycle superrigidity theorem for nonelementary cocycles $G \times X \to H$ with values in a closed subgroup $H < \mathrm{Isom}(X)$ of the isometry group of a 'negatively curved' space. All such groups are in class $\mathcal{S}$ by Corollary 3.4.10. It is therefore not surprising that one can also prove a cocycle superrigidity theorem for cocycles with values in a group H satisfying property (S). We do this in Theorem 4.4.1.

Applying cocycle superrigidity to the cocycles given by a stable orbit equivalence between essentially free, irreducible pmp actions $G_1 \times G_2 \curvearrowright (X, \mu)$ and $H_1 \times H_2 \curvearrowright (Y, \eta)$, we obtain the following orbit equivalence strong rigidity theorem (see Theorem 4.4.2): if G_1 and G_2 are nonamenable, while H_1 and H_2 are nonamenable and in class $\mathcal{S}$, then the actions must be conjugate.

Again, such an orbit equivalence strong rigidity theorem should not come as a surprise: in [Sak09b, Theorem 40], Sako proved exactly this result when G_1, G_2 and H_1, H_2 are *countable* groups in class $\mathcal{S}$. However, he does not use or prove a cocycle superrigidity theorem. The main novelty of this section is that our approach is surprisingly simple and short.

Using this two results, together with the result on uniqueness of Cartan subalgebras (F), we finally prove a W*-strong rigidity result for such product groups (see Theorem 4.4.4): if G_1 and G_2 are nonamenable, while H_1 and H_2 are nonamenable, weakly amenable and in class $\mathcal{S}$, then an isomorphism of the crossed products implies conjugacy of the actions.

Given locally compact groups G and H and a nonsingular action $G \curvearrowright (X, \mu)$, a Borel cocycle $\omega : G \times X \to H$ is a Borel map satisfying

$$\omega(gh, x) = \omega(g, h \cdot x)\, \omega(h, x)$$

for all $g, h \in G$ and $x \in X$. In a measurable context, the slightly more appropriate notion of cocycle is however the following. Denote by $\mathcal{M}(X, H)$ the Polish group of Borel functions from X to H, modulo functions equal almost everywhere. The group G acts continuously on $\mathcal{M}(X, H)$ by $(\alpha_g(F))(x) = F(g^{-1} \cdot x)$. Then a cocycle is a continuous map

$$\omega : G \to \mathcal{M}(X, H) : g \mapsto \omega_g$$

satisfying $\omega_{gh} = \alpha_h^{-1}(\omega_g)\omega_h$ for all $g, h \in G$. Every Borel cocycle ω gives rise to the cocycle $\omega_g = \omega(g, \cdot)$. Conversely, every cocycle can be realized by a Borel cocycle after removing from X a G-invariant Borel set of measure zero, see e.g. [Zim84, Theorem B.9].

The (measurable) cocycles ω and ω' are called cohomologous if there exists an element $\varphi \in \mathcal{M}(X, H)$ such that

$$\omega'_g = \alpha_{g^{-1}}(\varphi)\, \omega_g\, \varphi^{-1} \quad \text{for all } \ g \in G \ .$$

Borel cocycles $\omega, \omega' : G \times X \to H$ are called cohomologous if there exists a Borel map $\varphi : X \to H$ such that

$$\omega'(g, x) = \varphi(g \cdot x)\, \omega(g, x)\, \varphi(x)^{-1} \quad \text{for all } \ g \in G, x \in X \ .$$

Again, if two Borel cocycles are measurably cohomologous, then they also are Borel cohomologous on a conull G-invariant Borel set.

As in [MS04, Theorem 1.2], the following cocycle superrigidity theorem says that every "nonelementary" cocycle for an irreducible action $G_1 \times G_2 \curvearrowright (X, \mu)$ with values in a group with property (S) is cohomologous to a group homomorphism. In our context, being "nonelementary" is expressed by a non relative amenability property introduced in [Ana82] (see the discussion preceding Corollary 4.3.6).

Theorem 4.4.1. *Let G_1, G_2 and H be locally compact groups and $G_1 \times G_2 \curvearrowright (X, \mu)$ a pmp action with G_2 acting ergodically. Assume that H has property (S). Let $\omega : G_1 \times G_2 \times X \to H$ be a cocycle. Then at least one of the following statements holds.*

(i) *There exist closed subgroups $K \subseteq \widetilde{H} \subseteq H$ such that K is compact and $K \subseteq \widetilde{H}$ is normal, and there exists a continuous group homomorphism $\delta : G_1 \to \widetilde{H}/K$ with dense image such that ω is cohomologous to a cocycle ω_0 satisfying $\omega_0(g_1 g_2, x) \in \delta(g_1)K$ for all $g_i \in G_i$ and a.e. $x \in X$.*

(ii) *The action $G_1 \curvearrowright X \times H$ given by $g_1 \cdot (x, h) = \big(g_1 x, \omega(g_1, x)h\big)$ is amenable in the sense of Zimmer, i.e. there exists a G_1-equivariant conditional expectation $L^\infty(X \times H) \to L^\infty(X)$.*

Proof. Throughout the proof, we write $G = G_1 \times G_2$ and we view G_1 and G_2 as closed subgroups of G. We fix a left invariant Haar measure λ on H. We denote by $h \cdot \xi$ the left translation action of H on $L^2(H)$.

Formulation of the dichotomy. We are in precisely one of the following situations.

1. There exists no sequence $g_n \in G_2$ such that $\omega(g_n, \cdot) \to \infty$ in measure. More precisely, there exists a compact subset $L \subseteq H$ and an $\varepsilon > 0$ such that for all $g \in G_2$ the set $\{x \in X \mid \omega(g, x) \in L\}$ has measure at least ε.
2. There exists a sequence $g_n \in G_2$ such that $\omega(g_n, \cdot) \to \infty$ in measure.

Case 1. Fix such a compact set $L \subseteq H$ and $\varepsilon > 0$. Define the unitary representation

$$\pi : G \to \mathcal{U}(L^2(X \times H)) : (\pi(g)^* \xi)(x, h) = \xi(g \cdot x, \omega(g, x)h)$$

for all $g \in G$, $x \in X$, $h \in H$ and $\xi \in L^2(X \times H)$. Fix a compact subset $L_0 \subseteq H$ with $\lambda(L_0) > 0$. Given a Borel set $A \subseteq H$ of finite measure, denote by $1_A \in L^2(H)$ the characteristic function of A. By our choice of L and ε, we find that

$$\langle \pi(g)^*(1 \otimes 1_{LL_0}), 1 \otimes 1_{L_0} \rangle \geqslant \mu(\{x \in X \mid \omega(g,x) \in L\})\lambda(L_0) \geqslant \varepsilon\, \lambda(L_0)$$

for all $g \in G_2$. It follows that the vector $\xi \in L^2(X \times H)$ of minimal norm in the closed convex hull of $\{\pi(g)(1 \otimes 1_{LL_0}) \mid g \in G_2\}$ is nonzero. Clearly $\pi(g)\xi = \xi$ for $g \in G_2$. Viewing ξ as a Borel map $\xi : X \to L^2(G)$, we have

$$\xi(g_2 \cdot x) = \omega(g_2, x) \cdot \xi(x)$$

for all $g_2 \in G_2$ and a.e. $x \in X$. Since $x \mapsto \|\xi(x)\|_2$ is essentially G_2-invariant and $G_2 \curvearrowright (X, \mu)$ is ergodic, we have $\|\xi(x)\|_2 = c > 0$ for some $c \in \mathbb{R}_0^+$ and a.e. $x \in X$. After renormalization, we may assume that $c = 1$.

Denote by $T = \{\xi_0 \in L^2(H) \mid \|\xi_0\|_2 = 1\}$ the unit sphere of $L^2(H)$. The left translation action $H \curvearrowright T$ has closed orbits and thus $H \backslash T$ is a well defined Polish space. Since the map $x \mapsto H \cdot \xi(x)$ from X to $H \backslash T$ is G_2-invariant, it is constant a.e. So, we find a unit vector $\xi_0 \in L^2(H)$ satisfying $H \cdot \xi_0 = H \cdot \xi(x)$ for a.e. $x \in X$. Define the closed subgroup $K \subseteq H$ given by

$$K = \mathrm{Stab}(\xi_0) = \{s \in H \mid s \cdot \xi_0 = \xi_0\} \ .$$

Note that since $\xi_0 \in L^2(H)$, we must have that K is compact. Using a Borel inverse of the Borel bijection

$$H/K \to L^2(H) : hK \mapsto h\xi_0,$$

together with [Kec95, Theorem 12.17], we find a Borel map $\varphi : X \to H$ such that $\xi(x) = \varphi(x) \cdot \xi_0$ for a.e. $x \in X$. Replacing ω by the cohomologous cocycle given by

$$(g, x) \mapsto \varphi(g \cdot x)^{-1}\, \omega(g, x)\, \varphi(x) \ ,$$

we find that $\omega(g_2, x) \cdot \xi_0 = \xi_0$ for all $g_2 \in G_2$ and a.e. $x \in X$ and hence $\omega(g_2, x) \in K$ for all $g_2 \in G_2$ and a.e. $x \in X$.

For every cocycle $\alpha : G_2 \times X \to K$, we denote by $K_\alpha \subseteq K$ the subgroup generated by the elements $\alpha(g, x)$ for $g \in G$ and $x \in X$. Following [Zim76, Definition 3.7], we call a cocycle $\alpha : G_2 \times X \to K$ minimal if there is no cocycle $\beta : G_2 \times X \to K$ such that $K_\beta \subsetneq K_\alpha$. By [Zim76, Corollary 3.8], we can replace ω by a cohomologous cocycle and such that the restriction $\omega_2 = \omega|_{G_2 \times X}$ is minimal. After shrinking K to the subgroup generated by $\{\omega(g, x)\}_{g \in G_2, x \in X}$, [Zim76, Corollary 3.8] yields that the associated action $G_2 \curvearrowright X \times K$ given by

$$g_2 \cdot (x, k) = (g_2 x, \omega_2(g_2, x) k)$$

is ergodic.

Whenever $g_1 \in G_1$ and $g_2 \in G_2$, we have for a.e. $x \in X$

$$\omega(g_1, g_2x)\,\omega(g_2, x) = \omega(g_1g_2, x) = \omega(g_2g_1, x) = \omega(g_2, g_1x)\,\omega(g_1, x). \quad (4.4.1)$$

Fix $g_1 \in G_1$. It follows from (4.4.1) that for all $g_2 \in G_2$ and a.e. $x \in X$, $\omega(g_1, g_2x) \in K\omega(g_1, x)K$. Therefore, the map

$$X \to K\backslash H/K : x \mapsto K\omega(g_1, x)K$$

is G_2-invariant and thus constant a.e. Similarly as before, we find $s \in H$ and Borel maps $\varphi, \psi : X \to K$ such that (g_1 still being fixed) we have $\omega(g_1, x) = \varphi(x)\, s\, \psi(x)$ for a.e. $x \in X$.

Then (4.4.1) becomes

$$\varphi(g_2x)\, s\, \psi(g_2x)\, \omega(g_2, x) = \omega(g_2, g_1x)\, \varphi(x)\, s\, \psi(x)$$

for all $g_2 \in G_2$ and a.e. $x \in X$. So, the cocycle

$$\omega_2' : G_2 \times X \to K : (g_2, x) \mapsto \psi(g_2x)\, \omega_2(g_2, x)\, \psi(x)^{-1}$$

is cohomologous to ω_2 (as cocycles for $G_2 \curvearrowright X$ with values in the compact group K) and takes values in $K \cap s^{-1}Ks$. The minimality of ω_2 then implies that $K \cap s^{-1}Ks = K$. Similarly, the cocycle $\omega_3 : G_2 \times X \to K$ defined by $(g_2, x) \mapsto \omega(g_2, g_1x)$ is cohomologous to ω_2 by (4.4.1). Since ω_2 is minimal and $\omega_3(g_2, x) \in K = K_\omega$ for all $g_2 \in G_2$ and $x \in X$, we have that ω_3 is also minimal. Now, the cocycle

$$\omega_3' : G_2 \times X \to K : (g_2, x) \mapsto \varphi(g_2x)\, \omega_3(g_2, x)\, \varphi(x)^{-1}$$

is cohomologous to ω_3 and takes values in $K \cap sKs^{-1}$. The minimality of ω_3 then implies that $K \cap sKs^{-1} = K$. Defining the closed subgroup $H' \subseteq H$ given by

$$H' := \{s \in H \mid sKs^{-1} = K\}\,,$$

we find that $s \in H'$. By construction, K is a normal subgroup in H'. Moreover,

$$\omega(g_1, x) = \varphi(x)s\psi(x) \in KsK = sK$$

for a.e. $x \in X$.

We have proved that for every $g_1 \in G_1$, there exists an $s \in H'$ such that $\omega(g_1, x) \in sK$ for a.e. $x \in X$. We already had $\omega(g_2, x) \in K$ for all $g_2 \in G_2$ and a.e. $x \in X$. Hence, the Borel set

$$E = \{(g_1, g_2, x, y) \in G_1 \times G_2 \times X \times X \mid \omega(g_1g_2, x)K \neq \omega(g_1g_2, y)K\}\,.$$

is a null set. By Fubini, it follows that the complement of

$$F = \{x \in X \mid \omega(g_1g_2, x)K = \omega(g_1g_2, y)K \text{ for a.e. } y \in X, \text{ a.e. } g_i \in G_i\}$$

is a null set. Fixing $x_0 \in F$, we define $\delta_0 : G_1 \to H'/K$ by $\delta_0(g_1) = \omega(g_1, x_0)K$. Then, $\omega(g_1g_2, x) = \omega(g_1, g_2x)\omega(g_2, x) \in \delta_0(g_1)K$ for a.e. $g_1 \in G_1$, a.e. $g_2 \in G_2$ and a.e. $x \in X$. Since

$$\delta_0(g_1g_1') = \omega(g_1g_1', x_0)K = \omega(g_1, g_1'x_0)\omega(g_1', x_0)K = \delta_0(g_1)\delta_0(g_1')$$

for a.e. $g_1, g_1' \in G_1$ and a.e. $x \in X$, applying [Zim84, Theorem B.2] yields a Borel morphism $\delta : G_1 \to H'/K$ satisfying $\delta = \delta_0$ a.e. By [Zim84, Theorem B.3], δ is automatically continuous. Defining $\widetilde{H} < H'$ as the inverse image of the closure of $\delta(G_1)$, the first statement in the theorem holds.

Case 2. Fix a sequence $(g_n)_n$ in G_2 such that $\omega(g_n, \cdot) \to \infty$ in measure. Since H has property (S), we can take a map $\eta : H \to \mathcal{S}(H)$ as in Proposition 3.1.2 (ii). Define the sequence of Borel maps

$$\eta_n : X \to \mathcal{S}(H) : x \mapsto \eta(\omega(g_n, x)^{-1}) \; .$$

By (4.4.1), we have for all $g \in G_1$, a.e. $x \in X$ and all $n \in \mathbb{N}$ that

$$\omega(g_n, gx)^{-1} = \omega(g, x)\, \omega(g_n, x)^{-1}\, \omega(g, g_nx)^{-1} \; . \tag{4.4.2}$$

Fix $g \in G_1$ and fix $\varepsilon > 0$. Using Lemma 2.5.12, we take a compact subset $L \subseteq H$ such that $\omega(g, x) \in L$ for all x in a set of measure at least $1 - \varepsilon$. Then take a compact subset $L_1 \subseteq H$ such that

$$\|\eta(h_1hh_2^{-1}) - h_1 \cdot \eta(h)\|_1 < \varepsilon$$

for all $h_1, h_2 \in L$ and all $h \in H \setminus L_1$. Finally take n_0 such that for all $n \geqslant n_0$, we have that $\omega(g_n, x)^{-1} \in H \setminus L_1$ for all x in a set of measure at least $1 - \varepsilon$.

So, for our fixed $g \in G_1$ and for all $n \geqslant n_0$, there exists a Borel set $X_n \subseteq X$ of measure at least $1 - 3\varepsilon$ such that

$$\omega(g, x) \in L \;\; , \;\; \omega(g_n, x)^{-1} \in H \setminus L_1 \;\; , \;\; \omega(g, g_nx) \in L$$

for all $x \in X_n$. Applying η to (4.4.2), we conclude that for our fixed $g \in G_1$ and all $n \geqslant n_0$, we have

$$\|\eta_n(gx) - \omega(g, x) \cdot \eta_n(x)\|_1 < \varepsilon$$

for all $x \in X_n$. Since $\mu(X_n) \geqslant 1 - 3\varepsilon$, we have proved that for every $g \in G_1$, the sequence of functions

$$x \mapsto \|\eta_n(x) - \omega(g, x)^{-1} \cdot \eta_n(gx)\|_1 \tag{4.4.3}$$

converges to zero in measure.

Define the normal conditional expectations $P_n : L^\infty(X \times H) \to L^\infty(X)$ given by

$$\big(P_n(f)\big)(x) = \int_H f(x, h)\eta_n(x)(h)\,\mathrm{d}h$$

for all $f \in L^\infty(X \times H)$ and a.e. $x \in X$. Choose a point-weak* limit point $P : L^\infty(X \times H) \to L^\infty(X)$. We have

$$\begin{aligned}
&|\big(P_n(g_1 \cdot f)(x) - P_n(f)\big)(g_1^{-1}x)| \\
&\quad = \left| \int_H f\big(g_1^{-1}x, \omega(g_1^{-1}, x)h\big)\eta_n(x)(h)\,\mathrm{d}h - \int_H f(g_1^{-1}x, h)\eta_n(g_1^{-1}x)\,\mathrm{d}h \right| \\
&\quad \leqslant \int_H |f(g_1^{-1}x, h)| \left|\eta(x)(\omega(g_1^{-1}, x)^{-1}h) - \eta(g_1^{-1}x)(h)\right| \mathrm{d}h \\
&\quad \leqslant \|f\|_\infty \left\|\omega(g_1^{-1}, x) \cdot \eta(x) - \eta(g_1^{-1}, x)\right\|_1
\end{aligned}$$

for $n \in \mathbb{N}$, $f \in L^\infty(X \times H)$ and a.e. $x \in X$. By (4.4.3), it follows that $P_n(g_1 \cdot f) - g_1 \cdot P_n(f)$ converges to zero in the weak* topology. We conclude that P is a G_1-equivariant conditional expectation. So the second statement in the theorem holds. $\square$

Using the previous cocycle superrigidity theorem we prove the following orbit equivalence strong rigidity theorem. As mentioned above, for countable groups, the same result was obtained in [Sak09b, Theorem 40].

Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \eta)$ be essentially free, nonsingular actions of the locally compact groups G, H. Recall that we call these actions *stably orbit equivalent* if they admit cross sections such that the associated cross section equivalence relations are isomorphic. Recall that an action $G \curvearrowright (X, \mu)$ of a product group $G = G_1 \times G_2$ is called *irreducible* both actions $G_i \curvearrowright (X, \mu)$ are ergodic.

Theorem 4.4.2. *Let $G = G_1 \times G_2$ and $H = H_1 \times H_2$ be unimodular locally compact groups without nontrivial compact normal subgroups. Assume that $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \eta)$ are essentially free, irreducible, pmp actions. Assume that G_1, G_2 are nonamenable and that H_1 and H_2 have property (S).*

If the actions are stably orbit equivalent, then they must be conjugate.

More precisely, if (X_1, μ_1) *and* (Y_1, η_1) *are cross sections, with cross section equivalence relations* $\mathcal{R}$ *and* $\mathcal{S}$*, and if* $\pi : X_1 \to Y_1$ *is a nonsingular Borel isomorphism between the equivalence relations* $\mathcal{R}$ *and* $\mathcal{S}$*, then there exist conull* $\mathcal{R}$*-invariant (resp.* $\mathcal{S}$*-invariant) Borel sets* $X_2 \subseteq X_1$ *and* $Y_2 \subseteq Y_1$*, a Borel bijection* $\Delta : G \cdot X_2 \to H \cdot Y_2$ *and a continuous group isomorphism* $\delta : G \to H$ *such that*

- *(i)* $X_0 = G \cdot X_2$ *and* $Y_0 = H \cdot Y_2$ *are conull Borel sets and* $\Delta_*(\mu) = \eta$*,*
- *(ii)* $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ *for all* $g \in G$ *and all* $x \in X_0$*,*
- *(iii)* $\Delta(x) \in H \cdot \pi(x)$ *for all* $x \in X_2$*,*
- *(iv)* δ *is either of the form* $\delta_1 \times \delta_2$ *where* $\delta_i : G_i \to H_i$ *are isomorphisms, or of the form* $(g_1, g_2) \mapsto \big(\delta_2(g_2), \delta_1(g_1)\big)$ *where* $\delta_1 : G_1 \to H_2$ *and* $\delta_2 : G_2 \to H_1$ *are isomorphisms,*
- *(v)* *normalizing the Haar measures* λ_G *and* λ_H *such that* $\delta_*(\lambda_G) = \lambda_H$*, we have* $\mathrm{covol}(X_1) = \mathrm{covol}(Y_1)$*.*

Before starting the proof of this theorem, we prove the following lemma.

Lemma 4.4.3. *Let* G*,* H_1 *and* H_2 *be a locally compact group. Suppose that* G *is nonamenable and that* H_1 *and* H_2 *have property (S). Write* $H = H_1 \times H_2$*. Assume that* $G \curvearrowright (X, \mu)$ *and* $H \curvearrowright (Y, \eta)$ *are essentially free, irreducible, pmp actions. For* $i = 1, 2$*, let* $\omega_i : G \times X \to H_i$ *be cocycles and consider the actions* $G \curvearrowright X \times H_i$ *defined by*

$$g \cdot (x, h_i) = \big(gx,\, \omega_i(g, x)h_i\big).$$

If there exists a G*-equivariant positive linear map* $\Theta : L^\infty(G) \to L^\infty(X \times H)$*, then there is at most one* $i \in \{1, 2\}$ *for which there exists a* G*-equivariant conditional expectation* $L^\infty(X \times H_i) \to L^\infty(X)$*.*

Proof. Suppose that such G-equivariant conditional expectations $E_i : L^\infty(X \times H_i) \to L^\infty(X)$ exist for both $i = 1, 2$. Identifying $L^\infty(X \times H) \cong L^\infty\big(H_1; L^\infty(X \times H)\big)$ and $L^\infty(X \times H_1) \cong L^\infty\big(H_1; L^\infty(X)\big)$ as in Example 2.4.46, we define

$$\widetilde{E}_2 : L^\infty(X \times H) \to L^\infty(X \times H_1) : (\widetilde{E}_2 f)(h_1) = E_2\big(f(h_1)\big).$$

Defining the cocycle $\omega : G \to H$ by $\omega(g, x) = \big(\omega_1(g, x), \omega_2(g, x)\big)$ for $g \in G$ and $x \in X$, we have that $E = E_1 \circ \widetilde{E}_2$ is a G-equivariant conditional expectation

$L^\infty(X \times H_1 \times H_2) \to L^\infty(X)$ w.r.t. the action $g \cdot (x,h) = (g \cdot x, \omega(g,x)h)$. Composing with the G-invariant probability measure μ on X, we find a G-invariant state on $L^\infty(X \times H)$. Composing Θ with this state, it follows that G is amenable, contrary to our assumptions. □

We are now ready to prove Theorem 4.4.2

Proof of Theorem 4.4.2. Replacing X and Y by a conull G-invariant, resp. H-invariant, Borel set, we may assume that the Borel actions $G \curvearrowright X$ and $H \curvearrowright Y$ are free. Moreover, since $G \cdot X_1$ and $H \cdot Y_1$ are conull and Borel, we may assume that $X = G \cdot X_1$ and $Y = H \cdot Y_1$.

Let μ_1 (resp. η_1) be the canonical $\mathcal{R}$-invariant (resp. $\mathcal{S}$-invariant) probability measure on X_1 (resp. Y_1) from Proposition 2.5.38 and Remark 2.5.39. Normalize the Haar measures λ_G and λ_H such that $\operatorname{covol}(X_1) = 1 = \operatorname{covol}(Y_1)$. Take compact neighborhoods $\mathcal{U}$ of e in G and $\mathcal{V}$ of e in H such that the maps

$$\Psi : \mathcal{U} \times X_1 \to X : (k,x) \mapsto k \cdot x \quad \text{and} \quad \Phi : \mathcal{V} \times Y_1 \to Y : (\ell, y) \mapsto \ell \cdot y \tag{4.4.4}$$

are injective. By the definition of a cross section and its covolume (see Remark 2.5.39), these maps satisfy

$$\Psi_*\big((\lambda_G)|_{\mathcal{U}} \times \mu_1\big) = \mu|_{\mathcal{U}\cdot X_1} \quad \text{and} \quad \Phi_*\big((\lambda_H)|_{\mathcal{V}} \times \eta_1\big) = \eta|_{\mathcal{V}\cdot Y_1} \ .$$

Since $\mathcal{S}$ is ergodic, the $\mathcal{S}$-invariant probability measure η_1 is the unique. Since also $\pi_*(\mu_1)$ is $\mathcal{S}$-invariant, it follows that $\pi_*(\mu_1) = \eta_1$.

We start by translating the stable orbit equivalence π into a measure equivalence between G and H. This is quite standard and can also be found in [KKR17].

Choose a Borel map $p : X \to X_1$ such that $p(x) \in G \cdot x$ for all $x \in X$ and $p(kx) = x$ for all $k \in \mathcal{U}$, $x \in X_1$. Extend π to the Borel map $\rho : X \to Y$ defined by $\rho = \pi \circ p$. Similarly choose a Borel map $q : Y \to Y_1$ and define $\widetilde{\rho} : Y \to X : \widetilde{\rho} = \pi^{-1} \circ q$. By construction, $\rho(G \cdot x) \in H \cdot \rho(x)$ and $\widetilde{\rho}(H \cdot y) \in G \cdot \widetilde{\rho}(y)$ for all $x \in X$ and all $y \in Y$. Since the actions $G \curvearrowright X$ and $H \curvearrowright Y$ are free, we have unique Borel cocycles $\omega : G \times X \to H$ and $\zeta : H \times Y \to G$ such that

$$\rho(gx) = \omega(g,x) \cdot \rho(x) \qquad \text{and} \qquad \widetilde{\rho}(hy) = \zeta(h,y) \cdot \widetilde{\rho}(y)$$

for all $g \in G$, $h \in H$, $x \in X$ and $y \in Y$. Since $\widetilde{\rho}(\rho(x)) \in G \cdot x$ and $\rho(\widetilde{\rho}(y)) \in H \cdot y$ for all $x \in X$ and all $y \in Y$, we also have unique Borel maps $\varphi : X \to G$ and $\psi : Y \to H$ such that

$$\widetilde{\rho}(\rho(x)) = \varphi(x) \qquad \text{and} \qquad \rho(\widetilde{\rho}(y)) = \psi(y) \cdot y$$

for all $x \in X$ and $y \in Y$.

Define the measure preserving Borel actions $G \times H \curvearrowright X \times H$ and $G \times H \curvearrowright Y \times G$ given by

$$\begin{aligned} (g,h) \cdot (x,h') &= (gx, \omega(g,x)\, h'\, h^{-1}) \;, \\ (g,h) \cdot (y,g') &= (hy, \zeta(h,y)\, g'\, g^{-1}) \;. \end{aligned} \tag{4.4.5}$$

It is straightforward to check that

$$\theta : X \times H \to Y \times G : (x,h) \mapsto \big(h^{-1}\rho(x) \,,\, \zeta(h^{-1},\rho(x))\, \varphi(x)\big) \tag{4.4.6}$$

is a $G \times H$-equivariant Borel map and that θ is a bijection with inverse

$$\theta^{-1} : Y \times G \to X \times H : \theta^{-1}(y,g) = \big(g^{-1}\widetilde{\rho}(y) \,,\, \omega(g^{-1},\widetilde{\rho}(y))\, \psi(y)\big) \;.$$

Using the maps Ψ and Φ given by (4.4.4), one checks that

$$\theta(\Psi(k,x),\ell^{-1}) = (\Phi(\ell,\pi(x)),k^{-1})$$

for $k \in \mathcal{U}$, $\ell \in \mathcal{V}$ and $x \in X_1$. Since Ψ, Φ and π are measure preserving, it follows that the restriction of θ to $\mathcal{U} \cdot X_1 \times \mathcal{V}^{-1}$ is measure preserving. Since the actions of $G \times H$ on $X \times H$ and $Y \times G$ are measure preserving and since the map θ is $G \times H$-equivariant, it follows that the entire map θ is measure preserving.

The main part of the proof consists of using Theorem 4.4.1 to show that the cocycle ω is cohomologous to an isomorphism of groups $\delta : G \to H$. Write $\omega(g,x) = (\omega_1(g,x), \omega_2(g,x)) \in H_1 \times H_2$.

The map $\Theta : L^\infty(G) \to L^\infty(X \times H)$ defined by

$$(\Theta f)(x,h) = f\big(\varphi(x)^{-1}\zeta(h,h\rho(x))\big)$$

for $f \in L^\infty(G)$ and $x \in X$ and $h \in H$ is G-equivariant by (4.4.6). Hence, by Lemma 4.4.3, we find an $i \in \{1,2\}$ for which there exists no G_1-equivariant conditional expectation $L^\infty(X \times H_1) \to L^\infty(X)$ w.r.t. the actions $G_1 \curvearrowright X \times H_i$ given by $g \cdot (x,h_i) = (gx, \omega(g,x)h_i)$. Assume that there is no G_1-equivariant conditional expectation $L^\infty(X \times H_1) \to L^\infty(X)$. We prove that the conclusions of the theorem hold with the group isomorphism $\delta : G \to H$ being of the form $\delta_1 \times \delta_2$. In the case where there is no G_1-equivariant conditional expectation $L^\infty(X \times H_2) \to L^\infty(X)$, we exchange the roles of H_1 and H_2 and obtain again that the conclusions of the theorem hold with δ being of the form $(g_1,g_2) \mapsto (\delta_2(g_2), \delta_1(g_1))$.

Applying Theorem 4.4.1 to the cocycle ω_1, we find a compact subgroup $K_1 \subseteq H_1$, a closed subgroup $\widetilde{H}_1 \subseteq H_1$ with K_1 being a normal subgroup of $\widetilde{H}_1$, and a continuous group homomorphism $\delta_1 : G_1 \to \widetilde{H}_1/K_1$ with dense image such that ω_1 is cohomologous (as a measurable cocycle) to a cocycle $\widetilde{\omega}_1 : G \times X \to \widetilde{H}_1$ satisfying $\widetilde{\omega}_1(g_1 g_2, x) \in \delta_1(g_1)K_1$ for all $g_i \in G_i$ and a.e. $x \in X$.

In particular, there is an isomorphism between the actions of G on $X \times H_1$ induced by ω_1 and $\widetilde{\omega}_1$. Denote by $p : H_1 \to K_1\backslash H_1$ the projection. Fix a probability measure $\mu \in \mathrm{Prob}(K_1\backslash H_1)$. Then, $p_*\mu \in \mathrm{Prob}(H_1)$ is a K_1-invariant probability measure. Integrating w.r.t. $p_*\mu$ then yields a K_1-invariant state φ on $L^\infty(H_1)$. The map $E = \mathrm{id} \otimes \varphi : L^\infty(X \times H_1) \to L^\infty(X)$, then yields a G_2-equivariant conditional expectation w.r.t. the action induced by $\widetilde{\omega}_1$. Hence, composing E with the map $L^\infty(X \times H_1) \to L^\infty(X \times H_1)$ induced by isomorphism between the actions of G on $X \times H_1$ induced by ω_1 and $\widetilde{\omega}_1$, we get a G_2-equivariant conditional expectation $L^\infty(X \times H_1) \to L^\infty(X)$ w.r.t. the action induced by ω_1. Applying Lemma 4.4.3 to G_2 instead of G_1, it follows that there is no G_2-equivariant conditional expectation $L^\infty(X \times H_2) \to L^\infty(X)$ w.r.t. the action induced by ω_2.

We can again apply Theorem 4.4.1 and altogether, we find compact subgroups $K_i < H_i$, closed subgroups $\widetilde{H}_i < H_i$ with K_i being a normal subgroup of $\widetilde{H}_i$, and continuous group homomorphisms $\delta_i : G_i \to \widetilde{H}_i/K_i$ with dense image such that, writing $\delta = \delta_1 \times \delta_2$, $K = K_1 \times K_2$ and $\widetilde{H} = \widetilde{H}_1 \times \widetilde{H}_2$, we get that the cocycle ω is cohomologous (as a measurable cocycle) with a cocycle $\widetilde{\omega} : G \times X \to \widetilde{H}$ satisfying $\widetilde{\omega}(g, x) \in \delta(g)K$ for all $g \in G$ and a.e. $x \in X$.

The map $q : X \times H \to H/\widetilde{H} : (h, x) \mapsto h^{-1}\widetilde{H}$ induces an embedding

$$q^* : L^\infty(H/\widetilde{H}) \to L^\infty(X \times H) : f \mapsto f \circ q$$

that is H-equivariant w.r.t. the action $H \curvearrowright H/\widetilde{H}$ by left translation and $H \curvearrowright X \times H$ by right translation on the second component. Note that $q_*(f)$ is invariant for the action $G \curvearrowright X \times H$ induced by $\widetilde{\omega}$. Since ω is cohomologous to $\widetilde{\omega}$, we find a isomorphism $\varphi : X \times H \to X \times H$ between the actions $G \curvearrowright X \times H$ induced by $\widetilde{\omega}$ and ω respectively. Note that φ is moreover H-equivariant for the H-action by right translation on the second component. Together with q^*, this yields an H-equivariant embedding $L^\infty(H/\widetilde{H}) \to L^\infty(X \times H)^G$ where $H \curvearrowright X \times H$ is given by right translation on the second component and $G \curvearrowright X \times H$ is the action induced by ω. Using θ, we also find such an embedding into $L^\infty(Y \times G)^G = L^\infty(Y)$. Since the elements of $L^\infty(H_1/\widetilde{H}_1) \otimes 1 \subseteq L^\infty(H/\widetilde{H})$ are H_2-invariant, the irreducibility of the action $H \curvearrowright (Y, \eta)$ implies that $\widetilde{H}_1 = H_1$. Indeed, the only H_2-invariant functions in $L^\infty(Y)$ are a.e. constant, and hence by the embedding we constructed, so are all H_2-invariant functions in $L^\infty(H_1/\widetilde{H}_1)$. Similarly, we find that $\widetilde{H}_2 = H_2$. Since we assumed that the groups H_i have no nontrivial compact normal subgroups, we also conclude that K is trivial.

We have proved that ω is cohomologous, as a measurable cocycle, to the continuous group homomorphism $\delta : G \to H$ having dense image. We now prove that δ is bijective. Consider the unitary representation Π of G on $L^2(X \times H)$ given by $(\Pi(g)\xi)(x,h) = \xi(g^{-1}x, \delta(g)^{-1}h)$. Combining the map θ and the fact that the cocycle ω is cohomologous with δ, the representation Π is unitarily conjugate to the representation of G on $L^2(Y \times G)$ given by $(g{\cdot}\xi)(y,g') = \xi(y, g'g^{-1})$. Therefore, Π is a multiple of the regular representation. In particular, the coefficients

$$g \mapsto \langle \Pi(g)\xi, \eta\rangle$$

for $\xi, \eta \in L^2(X \times H)$ are in $C_0(G)$. In particular, for all compact subsets $D, E \subseteq H$, the function

$$f_{D,E} : G \to \mathbb{C} : g \mapsto\mapsto \langle \Pi(g)(1 \otimes 1_D), 1 \otimes 1_E\rangle = \lambda_H(\delta(g)D \cap E)$$

is in $C_0(G)$. It follows that δ is proper in the sense that the preimage of any compact is compact. Indeed, for every compact set $D \subseteq H$ with nonempty interior, we have

$$f_{D,D^2}(g) = \lambda_H(D) > 0$$

for all $g \in \delta^{-1}(D)$. Hence, if $\delta^{-1}(D)$ is not compact (and hence not contained in any compact since $\delta^{-1}(D)$ is closed), it follows that f_{D,D^2} is not in $C_0(G)$, which is a contradiction. In particular, $\ker \delta = \delta^{-1}(e)$ is compact, and since by assumption G has no nontrivial compact normal subgroups, we have $\ker \delta = \{e\}$. Since δ is proper, the image of δ is closed. Since δ has dense image, we conclude that δ is surjective. So we have proved that δ is bijective.

Since the Borel cocycles ω and δ are cohomologous as measurable cocycles, they are also cohomologous as Borel cocycles on a conull G-invariant Borel set $X_0 \subseteq X$. Since $\theta(X_0 \times H)$ is a conull $G \times H$-invariant Borel subset of $Y \times G$, it must be of the form $Y_0 \times H$. So we can restrict everything to X_0 and Y_0, and assume that $X_0 = X$ and $Y_0 = Y$. Choose a Borel map $\gamma : X \to H$ such that $\omega(g,x) = \gamma(gx)\,\delta(g)\,\gamma(x)^{-1}$ for all $g \in G$ and $x \in X$. Define the measure preserving Borel bijections $\theta_1, \theta_2 : X \times H \to X \times H$ given by

$$\theta_1(x,h) = (x, \gamma(x)h) \quad \text{and} \quad \theta_2(x,h) = (\delta^{-1}(h^{-1})x, h^{-1}) \; .$$

Write $\widetilde{\theta} = \theta \circ \theta_1 \circ \theta_2$. Consider the measure preserving Borel action of $G \times H$ on $X \times H$ given by

$$(g,h) \cdot (x,h') = (\delta^{-1}(h)x \, , \, h\,h'\,\delta(g)^{-1}) \; .$$

Still using the action of $G \times H$ on $Y \times G$ defined in (4.4.5), we get that $\widetilde{\theta}$ is $G \times H$-equivariant. Define the Borel functions $\Delta : X \to Y$ and $\widetilde{\gamma} : X \to G$ such that $\widetilde{\theta}(x,e) = (\Delta(x), \widetilde{\gamma}(x))$ for all $x \in X$. Then,

$$\widetilde{\theta}(x,h) = \widetilde{\theta}\big((e, \delta^{-1}(h^{-1})) \cdot (x,e)\big)$$

$$= (e, \delta^{-1}(h^{-1})) \cdot \widetilde{\theta}(x, e) \tag{4.4.7}$$
$$= (\Delta(x), \widetilde{\gamma}(x)\, \delta^{-1}(h))$$

for all $x \in X$ and $h \in H$. Since $\widetilde{\theta}$ is bijective and δ is bijective, also $\Delta : X \to Y$ must be bijective. Since $\widetilde{\theta}$ is H-equivariant, we have $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ for all $x \in X$.

Since $\widetilde{\theta}$ is measure preserving and δ is measure scaling, by (4.4.7), Δ must be measure scaling. Since both μ and η are probability measures, it follows that Δ is measure preserving and thus also δ that is measure preserving. By construction, $\Delta(x) \in H \cdot \rho(x)$ for all $x \in X$ and thus $\Delta(x) \in H \cdot \pi(x)$ for all $x \in X_1$. This concludes the proof of the theorem. □

We can now prove Theorem G.

Theorem 4.4.4. *Let $G = G_1 \times G_2$ and $H = H_1 \times H_2$ be unimodular locally compact groups without nontrivial compact normal subgroups. Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \eta)$ be essentially free, irreducible, pmp actions. Assume that G_1, G_2, H_1, H_2 are nonamenable and that H_1, H_2 are weakly amenable and in class $\mathcal{S}$.*

If $p(L^\infty(X) \rtimes G)p \cong q(L^\infty(Y) \rtimes H)q$ for nonzero projections p and q, then the actions are conjugate: there exists a continuous group isomorphism $\delta : G \to H$ and a pmp isomorphism $\Delta : X \to Y$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ for all $g \in G$ and a.e. $x \in X$.

Proof. Denote $M = L^\infty(X) \rtimes G$ and $N = L^\infty(Y) \rtimes H$. Take cross sections $X_1 \subseteq X$ and $Y_1 \subseteq Y$ for $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ respectively. Denote by $\mathcal{R}$ and $\mathcal{S}$ the associated cross section equivalence relations. By Proposition 2.5.41, we have

$$p_1 M p_1 \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big) \quad \text{and} \quad q_1 N q_1 \cong L(\mathcal{S}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{V})\big), \tag{4.4.8}$$

where $\mathcal{U} \subseteq G$ and $\mathcal{V} \subseteq H$ are neighborhoods of the identity for which the action maps $\mathcal{U} \times X_1 \to X$ and $\mathcal{V} \times Y_1 \to Y$ are injective and have nonnegligible range, and $p_1 = 1_{\mathcal{U} \cdot X_1}$ and $q_1 = 1_{\mathcal{V} \cdot Y_1}$.

Since M and N are both factors, the isomorphism $pMp \cong qNq$ together with (4.4.8) implies that

$$L(\mathcal{R}) \mathbin{\overline{\otimes}} B\big(\ell^2(\mathbb{N})\big) \cong L(\mathcal{S}) \mathbin{\overline{\otimes}} B\big(\ell^2(\mathbb{N})\big)$$

Denote by $\rho : L(\mathcal{R}) \mathbin{\overline{\otimes}} B\big(\ell^2(\mathbb{N})\big) \to L(\mathcal{S}) \mathbin{\overline{\otimes}} B\big(\ell^2(\mathbb{N})\big)$ the isomorphism. Denote by $A = L^\infty(X_1) \otimes \ell^\infty(\mathbb{N})$ and $B = L^\infty(Y_1) \otimes \ell^\infty(\mathbb{N})$ the Cartan subalgebras. By

Theorem F the von Neumann algebra N has unique Cartan subalgebra up to unitary conjugacy and by Proposition 2.5.30 so does $L(\mathcal{S}) \mathbin{\overline{\otimes}} B\big(\ell^2(\mathbb{N})\big)$. Hence, after conjugating with a unitary, we can assume $\pi(A) = B$. We find projections $p' \in A$ and $q' \in B$ with $p' \leqslant 1 \otimes e_{11}$ and $q' \leqslant 1 \otimes e_{nn}$ for some $n \in \mathbb{N}$ with $q' = \pi(p')$. Here, $e_{ij} \in B\big(\ell^2(\mathbb{N})\big)$ denote the matrix units.

Viewing p' and q' as projections in $L^\infty(X_1) = L(\mathcal{R})$ and $L^\infty(Y_1) = L(\mathcal{S})$ we can take measurable subsets $X_2 \subseteq X_1$ and $Y_2 \subseteq Y_1$ such that $p' = 1_{X_2}$ and $q' = 1_{Y_2}$. By ergodicity of the actions, both X_2 and Y_2 are also cross sections with cross section equivalence relations $\mathcal{R}' = \mathcal{R}|_{X_2}$ and $\mathcal{S}' = \mathcal{S}|_{Y_2}$ Identifying $L(\mathcal{R}') \cong p'Mp'$ and $L(\mathcal{S}') \cong q'Nq'$, the restricted isomorphism $\pi : L(\mathcal{R}') \to L(\mathcal{S}')$ satisfies $\pi\big(L^\infty(X_2)\big) = L^\infty(Y_2)$. Hence, $\mathcal{R}' \cong \mathcal{S}'$. Now, Theorem 4.4.2 concludes the theorem. □

4.5 A dichotomy for the stable normalizer

In this section, we prove a dichotomy result for the stable normalizer $\mathcal{N}^{\mathrm{s}}_M(A)$ similar to Theorem 4.2.2. In order to do that, we strengthen the weak amenability requirement in Theorem 4.2.2 to CMAP. Roughly speaking, the result then follows by adapting the methods in the proof of [BHV18, Proposition 3.6] to the abstract setting used in 4.2.2.

Recall that the stable normalizer of a von Neumann subalgebra $A \subseteq M$ is defined as

$$\mathcal{N}^{\mathrm{s}}_M(A) = \{x \in M \mid xAx^* \subseteq A \text{ and } x^*Ax \subseteq A\}.$$

The result that we obtained is now as follows.

Theorem 4.5.1. *Let G be a locally compact group in class $\mathcal{S}$ with CMAP. Let (M, Tr) be a von Neumann algebra with a faithful normal semifinite trace and $\Phi : M \to M \mathbin{\overline{\otimes}} L(G)$ a coaction. Let $p \in M$ be a projection with $\mathrm{Tr}(p) < \infty$ and $A \subseteq pMp$ a von Neumann subalgebra.*

If A is Φ-amenable then at least one of the following statements holds: A can be Φ-embedded or $\mathcal{N}^s_{pMp}(A)''$ stays Φ-amenable.

Proof. We start by making a first simplification. We replace M by $B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} M$ and define the projection $e = 1 \otimes 1 \otimes p$ in M. We then replace A by $\ell^\infty(\mathbb{N}) \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} A$ and view it as a von Neumann subalgebra of eMe. We finally replace p by the finite trace projection $e_{00} \otimes e_{00} \otimes p$ and the coaction Φ by $\mathrm{id} \otimes \mathrm{id} \otimes \Phi$. We are now in the following situation: M is a von Neumann algebra with a faithful normal semifinite trace Tr, $e \in M$ is a projection, $A \subseteq eMe$ is a von Neumann subalgebra with $\mathrm{Tr}|_A$ being semifinite and $p \in A$ is a projection of finite trace. By [BHV18, Lemma 3.5], for every $x \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$, there exist $u \in \mathcal{N}_{eMe}(A)$ and $a, b \in pAp$ such that $ua = x = bu$. In particular, $\mathcal{N}^{\mathrm{s}}_{pMp}(pAp)'' = p\mathcal{N}_{eMe}(A)''p$. Also, for every partial isometry $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$ with $v^*v = s$ and $vv^* = t$, there exists a $u \in \mathcal{N}_{eMe}(A)$ such that $us = v = tu$.

So, assume that pAp is Φ-amenable. We have to prove that pAp can be Φ-embedded or that $\mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$ is Φ-amenable. Write $q = \Phi(p)$ and $f = \Phi(e)$.

Step 1. If $u \in \mathcal{N}_{eMe}(A)$, then $p(A \cup \{u\})''p$ is still Φ-amenable.

Denote by $E : eMe \to A$ the unique Tr-preserving normal conditional expectation. Since pAp is Φ-amenable, by Theorem 2.4.64, there exists a conditional expectation

$$P : q\big(M \mathbin{\overline{\otimes}} B(L^2(G))\big)q \to \Phi(pAp)$$

satisfying $P(\Phi(x)) = \Phi(E(x))$ for all $x \in pMp$. We have a canonical isomorphism

$$f(M \mathbin{\overline{\otimes}} B(L^2(G)))f \cong B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} q(M \mathbin{\overline{\otimes}} B(L^2(G)))q$$

sending $\Phi(A)$ onto $\ell^\infty(\mathbb{N}) \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} \Phi(pAp)$. Denoting by $E_0 : B(\ell^2(\mathbb{N})) \to \ell^\infty(\mathbb{N})$ the normal conditional expectation and taking $E_0 \otimes \mathrm{id} \otimes P$, we can extend P to a conditional expectation

$$P : f(M \mathbin{\overline{\otimes}} B(L^2(G)))f \to \Phi(A)$$

satisfying $P(\Phi(x)) = \Phi(E(x))$ for all $x \in eMe$.

For every $n \geqslant 1$, define

$$P_n : f(M \mathbin{\overline{\otimes}} B(L^2(G)))f \to \Phi(A) : T \mapsto \frac{1}{n} \sum_{k=1}^{n} \Phi(u^k) P\big(\Phi(u^{-k}) T \Phi(u^k)\big) \Phi(u^{-k}).$$

Note that every P_n is a conditional expectation satisfying $P_n(\Phi(x)) = \Phi(E(x))$ for all $x \in eMe$. Define

$$P_0 : f(M \mathbin{\overline{\otimes}} B(L^2(G)))f \to \Phi(A)$$

as a point-w.o. limit point of the sequence $(P_n)_{n \geqslant 1}$. Then P_0 is a conditional expectation satisfying $P_0(\Phi(x)) = \Phi(E(x))$ for all $x \in eMe$ and

$$P_0(\Phi(u^k) T \Phi(u^{-k})) = \Phi(u^k) P_0(T) \Phi(u^{-k})$$

for all $T \in f\big(M \mathbin{\overline{\otimes}} B(L^2(G))\big)f$ and $k \in \mathbb{Z}$. Define the positive functional Ω_0 on $q(M \mathbin{\overline{\otimes}} B(L^2(G)))q$ given by $\Omega_0(T) = \mathrm{Tr}(\Phi^{-1}(P_0(T)))$, which is well defined because $P_0(T) \in \Phi(pAp)$. By construction, Ω_0 is $\Phi(pAp)$-central and $\Omega_0(\Phi(x)) = \mathrm{Tr}(x)$ for all $x \in pMp$.

Let $k \in \mathbb{Z}$ and $a \in A$. Put $x_0 = pu^k ap$. Note that $x_0 = u^k bp$, where $b \in A$ is defined as $b = u^{-k} pu^k a$. Using the notation $P' = \Phi^{-1} \circ P_0$, we have for every $T \in q(M \mathbin{\overline{\otimes}} B(L^2(G)))q$ that

$$\begin{aligned}\Omega_0(\Phi(x_0)T) &= \Omega_0\big(\Phi(u^k bp)T\big) = \mathrm{Tr}\left(P'(\Phi(u^k)\Phi(bp)T)\right)\\ &= \mathrm{Tr}\left(u^k bpP'(T\Phi(u^k))u^{-k}\right) = \mathrm{Tr}\left(P'(T\Phi(u^k))bp\right) \qquad (4.5.1)\\ &= \mathrm{Tr}\left(P'(T\Phi(u^k bp))\right) = \Omega_0\big(T\Phi(x_0)\big).\end{aligned}$$

By the bicommutant theorem (see Theorem 2.4.2) the $*$-subalgebra $\{pu^k ap \mid k \in \mathbb{Z}, a \in A\}$ is s.o. dense in $p(A \cup \{u\})''p$. Fix $T \in q(M \mathbin{\overline{\otimes}} B(L^2(G)))q$ and $x \in p(A \cup \{u\})''p$. Take a net $(x_i)_i$ in $\{pu^k ap \mid k \in \mathbb{Z}, a \in A\}$ such that $x_i \to x$ in s.o. topology. Using that $\Omega_0(\Phi(x)) = \mathrm{Tr}(x)$ and the Cauchy-Schwarz inequality, we have

$$|\Omega_0\big(\Phi(x_i)T\big) - \Omega_0\big(\Phi(x)T\big)| \leqslant \|x_i - x\|_{2,\mathrm{Tr}}\, \Omega_0(T^*T)$$

and hence

$$\Omega_0\big(\Phi(x)T\big) = \lim_i \Omega_0\big(\Phi(x_i)T\big)$$

whenever $(x_i)_i$ is a net in pMp converging to x in s.o. topology. Similarly,

$$\Omega_0\big(T\Phi(x)\big) = \lim_i \Omega_0\big(T\Phi(x_i)\big)$$

and hence by (4.5.1), we have $\Omega_0\big(\Phi(x)T\big) = \Omega_0\big(T\Phi(x)\big)$, meaning that Ω_0 is $\Phi\big(p(A \cup \{u\})''p\big)$-central. This concludes the proof of step 1.

Notations. Since G has CMAP, we can fix a sequence $\eta_n \in A(G)$ such that the associated completely bounded maps $m_n : L(G) \to L(G)$ and $\varphi_n : L(G) \to L(G)$ are as in Lemma 4.2.5. We denote $\psi_n : pMp \to pMp : \psi_n(x) = p\varphi_n(x)p$ the associated adapted maps.

Whenever $Q \subseteq eMe$ is a von Neumann subalgebra, we denote by $\mathcal{N}_Q$ the von Neumann subalgebra of $B\big(L^2(Me)\big) \mathbin{\overline{\otimes}} L(G)$ generated by $\Phi(M)$ and $\rho(Q)$, where $\rho(a)$ for $a \in Q$ is given by right multiplication in the first leg. We write $\mathcal{N} = \mathcal{N}_A$.

For every partial isometry $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$ with $s = v^*v$ and $t = vv^*$, denote by

$$\beta_v : \rho(s)\mathcal{N}\rho(s) \to \rho(t)\mathcal{N}\rho(t) : x \mapsto \rho(v^*)x\rho(v)$$

Note that $\beta_v(\Phi(x)\rho(a)) = \Phi(x)\rho(vav^*)$ for $x \in M$, $a \in A$ and $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$.

Define $q_1 = \Phi(p)\rho(p)$. We still denote by β_v the map $q_1\mathcal{N}q_1 \to q_1\mathcal{N}q_1$ given by $T \mapsto \beta_v(\rho(s)T\rho(s))$.

Step 2. Let $Q \subseteq eMe$ be a von Neumann subalgebra such that $A \subseteq Q$ and such that pQp is Φ-amenable. Then there exists a sequence of functionals $\mu_n^Q \in (q_1\mathcal{N}_Q q_1)_*$ with the following properties.

(a) $\mu_n^Q(\Phi(x)\rho(a)) = \mathrm{Tr}(\psi_n(x)a)$ for all $x \in pMp$ and $a \in pQp$.

(b) $\|\mu_n^Q\| \leqslant \mathrm{Tr}(p)$ for all n.

To prove step 2, by using Lemma 4.2.4 and Lemma 4.2.7, we get a sequence of completely positive maps $\psi_n : pMp \to pMp$ that are adapted with respect to Q and that satisfy $\|\psi_n\|_{\mathrm{adap}} \leqslant \Lambda_G \leqslant 1$ and such that $\psi_n(x) \to x$ in s.o. topology for $x \in pMp$. Moreover, we can assume that the associated maps $\theta_n : q_1\mathcal{N}q_1 \to B\big(L^2(pMp)\big)$ from Definition 4.2.6 are completely contractive. Now, composing with the vector functional $T \mapsto \langle T\hat{p}, \hat{p}\rangle$, which has norm $\mathrm{Tr}(p)$, the proof of step 1 is complete.

Step 3. The positive functionals $\omega_n = |\mu_n^A|$ in $(q_1\mathcal{N}q_1)_*$ satisfy

(a) $\lim_n \omega_n(\Phi(x)) = \mathrm{Tr}(x)$ for all $x \in pMp$,

(b) $\lim_n \omega_n(\Phi(a)\rho(a^*)) = \mathrm{Tr}(p)$ for all $a \in \mathcal{U}(pAp)$,

(c) for every partial isometry $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$, we have that $\lim_n \|\omega_n \circ \beta_{v^*} - \omega_n \circ \mathrm{Ad}\,\Phi(v)\| = 0$.

Note that, as defined above, the functional $\omega_n \circ \beta_{v^*}$ on $q_1\mathcal{N}q_1$ is given by $(\omega_n \circ \beta_{v^*})\big(\Phi(x)\rho(a)\big) = \omega_n\big(\Phi(x)\rho(v^*av)\big)$ for all $x \in pMp$ and $a \in pAp$, while the functional $\omega_n \circ \mathrm{Ad}\,\Phi(v)$ on $q_1\mathcal{N}q_1$ is given by $\big(\omega_n \circ \mathrm{Ad}\,\Phi(v)\big)\big(\Phi(x)\rho(a)\big) = \omega_n\big(\Phi(vxv^*)\rho(a)\big)$.

To prove step 3, let $Q \subseteq eMe$ be a von Neumann subalgebra such that $A \subseteq Q$ and such that pQp is Φ-amenable. Define μ_n^Q as in step 2 and denote $\omega_n^Q = |\mu_n^Q|$. Let $u_n \in q_1\mathcal{N}q_1$ be partial isometries such that $\mu_n^Q = u_n \cdot \omega_n^Q$ and such that $u_n^*u_n$ is the support projection of ω_n^Q (see Theorem 2.4.6). We have

$$\begin{aligned} |\mu_n^Q(T) - \omega_n^Q(T)|^2 &= |\omega_n^Q\big(T(u_n - 1)\big)|^2 \\ &\leqslant \omega_n^Q(T^*T)\omega_n^Q\big((u_n^* - 1)(u_n - 1)\big) \end{aligned} \tag{4.5.2}$$

$$\leqslant \left\|\omega_n^Q\right\| \|T\| \left(\omega_n^Q(1) - 2\operatorname{Re}(\mu_n^Q(1))\right)$$

for all $T \in q_1 \mathcal{N} q_1$. Since $\|\mu_n^Q\| \leqslant \operatorname{Tr}(p)$ for all n and $\lim_n \mu_n^Q(q_1) = \operatorname{Tr}(p)$, it follows that $\lim_n \|\mu_n^Q - \omega_n^Q\| = 0$. In particular, we have

$$\lim_n \omega_n^Q\big(\Phi(x)\big) = \lim_n \mu_n^Q\big(\Phi(x)\big) = \operatorname{Tr}\big(\psi_n(x)\big) = \operatorname{Tr}(x)$$

for all $x \in pMp$. Whenever $a \in \mathcal{U}(pQp)$, we get that $\lim_n \omega_n^Q(\Phi(a)\rho(a^*)) = \operatorname{Tr}(p)$. So the first two properties in step 3 are already proven. Similarly as in (4.5.2), one checks that

$$\lim_n \|\big(\Phi(a)\rho(a^*)\big)\cdot\omega_n^Q - \omega_n^Q\| = 0 = \lim_n \|\omega_n^Q \cdot \big(\Phi(a)\rho(a^*)\big) - \omega_n^Q\|$$

for all $a \in \mathcal{U}(pQp)$, where we write $\omega_n^Q \cdot u$ for the functional defined by $(\omega_n^Q \cdot u)(x) = \omega_n^Q(ux)$. Since every element $a \in pQp$ can be written as a sum of four unitaries, it follows that

$$\lim_n \|\Phi(a)\cdot\omega_n^Q - \rho(a)\cdot\omega_n^Q\| = 0 = \lim_n \|\omega_n^Q\cdot\Phi(a) - \omega_n^Q\cdot\rho(a)\| \tag{4.5.3}$$

for all $a \in pQp$.

To prove the third property in step 3, fix a partial isometry $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$ and write $s = v^*v$. Recall from above that we can take a $u \in \mathcal{N}_{eMe}(A)$ such that $v = us$. Define $Q = (A \cup \{u\})''$. By step 1 of the proof, pQp is Φ-amenable. By construction $v \in pQp$. Now, (4.5.3) yields that

$$\lim_n \|\omega_n^Q \circ \operatorname{Ad}\Phi(v) - \omega_n^Q \circ \beta_{v^*}\|$$

$$= \lim_n \left\|\Phi(v^*)\cdot\omega_n^Q - \rho(v^*)\cdot\omega_n^Q\right\| + \lim_n \left\|\omega_n^Q\cdot\Phi(v) - \omega_n^Q\cdot\rho(v)\right\| = 0$$

Since μ_n^Q and μ_n^A agree on the linear span of $\{\Phi(x)\rho(a)\}_{x\in pMp, a\in pQp}$, which is s.o. dense in $q_1\mathcal{N}q_1$, restricting to $q_1\mathcal{N}q_1$ yields the third property in step 3.

Notations. Choose a standard Hilbert space $\mathcal{H}$ for the von Neumann algebra $\mathcal{N}$, which comes with the normal $*$-homomorphism $\pi_\ell : \mathcal{N} \to B(\mathcal{H})$, the normal $*$-antihomomorphism $\pi_r : \mathcal{N} \to B(\mathcal{H})$ and the positive cone $\mathcal{H}^+ \subseteq \mathcal{H}$. As before, define for every $u \in \mathcal{N}_{eMe}(A)$ the automorphism β_u of $\mathcal{N}$ implemented by right multiplication with u^* on $L^2(Me) \otimes L^2(G)$ and denote by $W_u \in \mathcal{U}(\mathcal{H})$ its canonical implementation.

Denote by $E_{\mathcal{Z}} : pAp \to \mathcal{Z}(A)p$ the unique trace preserving conditional expectation (i.e. the center valued trace of pAp). For every projection $s \in pAp$, denote by $z_s \in \mathcal{Z}(A)p$ its central support, which equals the support projection of $E_{\mathcal{Z}}(s)$. Denote by $\mathcal{P}_0 \subseteq \mathcal{P}(pAp)$ the set of projections $s \in pAp$ for which there exists a $\delta > 0$ such that $E_{\mathcal{Z}}(s) \geqslant \delta z_s$. We then denote $D_s = (E_{\mathcal{Z}}(s))^{1/2}$ and we denote by D_s^{-1} the (bounded) inverse of D_s in $\mathcal{Z}(A)z_s$. As in [BHV18, Section 3] and using [BHV18, Lemma 3.9], we can choose a sequence $a_i \in pAp$ such that

$$\sum_{i=0}^{\infty} a_i a_i^* = D_s z_s \quad \text{and} \quad \sum_{i=0}^{\infty} a_i^* a_i = D_s^{-1} s \; .$$

We make once and for all such a choice for each $s \in \mathcal{P}_0$. We also define

$$T(s) = \sum_{i=0}^{\infty} \Phi(a_i)\rho(a_i^*) \in q_1 \mathcal{N} q_1 \; .$$

Note that the series defining $T(s)$ is convergent in s.o. topology, so that $T(s)$ is a well defined element of $q_1\mathcal{N}q_1$.

For every partial isometry $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$ with $s = v^*v$ and $t = vv^*$, we denote by $W_v : \pi_\ell(\rho(s))\pi_r(\rho(s))\mathcal{H} \to \pi_\ell(\rho(t))\pi_r(\rho(t))\mathcal{H}$ the canonical unitary implementation of the $*$-isomorphism $\beta_v : \rho(s)\mathcal{N}\rho(s) \to \rho(t)\mathcal{N}\rho(t)$.

Step 4. The canonical implementation $\xi_n \in \pi_\ell(q_1)\pi_r(q_1)\mathcal{H}$ of ω_n satisfies the following properties.

1. $\lim_n \langle \pi_\ell(\Phi(x))\xi_n, \xi_n \rangle = \mathrm{Tr}(pxp) = \lim_n \langle \pi_r(\Phi(x))\xi_n, \xi_n \rangle$ for all $x \in M$,
2. $\lim_n \|\pi_\ell(\Phi(a))\xi_n - \pi_r(\rho(a))\xi_n\| = 0$ for all $a \in \mathcal{U}(pAp)$,
3. Whenever $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$ is a partial isometry such that $s = v^*v$ and $t = vv^*$ belong to $\mathcal{P}_0$, we have

$$\lim_n \|\pi_\ell(\Phi(v))\xi_n - \pi_r(T(s)^*)\, \pi_r(\Phi(v))\, W_v^* \, \pi_r(T(t))\, \xi_n\| = 0 \; . \qquad (4.5.4)$$

The first two properties follow immediately from the first two properties of ω_n in step 3. To also deduce the third property from step 3, one can literally apply the proof of [BHV18, Proposition 3.6].

Notations and formulation of the dichotomy. As in the proof of Theorem 4.2.2, the coaction $\Psi : \mathcal{N} \to \mathcal{N} \mathbin{\overline{\otimes}} L(G)$ given by $\Psi = \mathrm{id} \otimes \Delta$ has a canonical unitary implementation $V \in B(\mathcal{H}) \mathbin{\overline{\otimes}} L(G)$ and an associated nondegenerate $*$-morphism $\pi : C_0(G) \to B(\mathcal{H})$. We again distinguish two cases.

- **Case 1.** For every $F \in C_0(G)$, we have that $\limsup_n \|\pi(F)\xi_n\| = 0$.

- **Case 2.** There exists an $F \in C_0(G)$ with $\limsup_n \|\pi(F)\xi_n\| > 0$.

We prove that in case 1, the von Neumann subalgebra $\mathcal{N}^{\mathrm{s}}_{pMp}(pAp)'' \subseteq pMp$ is Φ-amenable and that in case 2, the von Neumann subalgebra $pAp \subseteq pMp$ can be Φ-embedded.

The proof in case 2 is identical to the proof of case 2 in Theorem 4.2.2, because that part of the proof only relies on the first two properties of the net ξ_n in step 4. So from now on, assume that we are in case 1. Choose a positive functional Ω on $B(\mathcal{H})$ as a weak* limit point of the net of vector functionals $T \mapsto \langle T\xi_n, \xi_n\rangle$.

Denote $\mathcal{G} = \mathcal{N}_{eMe}(A)$. The group $\mathcal{G}$ acts on $\mathcal{N}$ by the automorphisms β_u, $u \in \mathcal{G}$. We also consider the diagonal action of $\mathcal{G}$ on $\mathcal{N} \otimes_{\mathrm{alg}} \mathcal{N}^{\mathrm{op}}$ and denote by D the algebraic crossed product $D = (\mathcal{N} \otimes_{\mathrm{alg}} \mathcal{N}^{\mathrm{op}}) \rtimes_{\mathrm{alg}} \mathcal{G}$. As a vector space, $D = \mathcal{N} \otimes_{\mathrm{alg}} \mathcal{N}^{\mathrm{op}} \otimes_{\mathrm{alg}} \mathbb{C}\mathcal{G}$ and the product and $*$-operation on D are given by

$$(x_1 \otimes y_1^{\mathrm{op}} \otimes u_1)\,(x_2 \otimes y_2^{\mathrm{op}} \otimes u_2) = x_1\beta_{u_1}(x_2) \otimes (\beta_{u_1}(y_2)y_1)^{\mathrm{op}} \otimes u_1u_2$$

and

$$(x \otimes y^{\mathrm{op}} \otimes u)^* = \beta_{u^*}(x^*) \otimes (\beta_{u^*}(y^*))^{\mathrm{op}} \otimes u^* \ .$$

We define the $*$-representations

$$\Theta : D \to B(\mathcal{H}) : x \otimes y^{\mathrm{op}} \otimes u \mapsto \pi_\ell(x)\,\pi_r(y)\,W_u \ \ ,$$

$$\Theta_1 : D \to B(\mathcal{H}) \mathbin{\overline{\otimes}} L(G) : x \otimes y^{\mathrm{op}} \otimes u \mapsto (\pi_\ell \otimes \mathrm{id})\Psi(x)\,(\pi_r(y)\,W_u \otimes 1) \ \ .$$

Define the $*$-subalgebras of $\mathcal{N}$ given by

$$\mathcal{N}_1 = \mathrm{span}\{\Phi(x)\rho(a)\}_{x\in M, a\in A},$$

$$\mathcal{N}_2 = \overline{\mathcal{N}_1},$$

and let $\mathcal{N}_3$ be the $*$-subalgebra of $\mathcal{N}$ given by all $x \in \mathcal{N}$ such that there exist sequences $(x_n)_n$ in M and $(a_n)_n$ in A with

$$x = \sum_n \Phi(x_n)\rho(a_n)$$

and such that $\sum_n x_n^*x_n$, $\sum_n x_nx_n^*$, $\sum_n a_n^*a_n$, $\sum_n a_na_n^*$ are bounded. Each $\mathcal{N}_i$ for $i = 1, 2, 3$ is globally invariant under the automorphisms β_u for $u \in \mathcal{G}$, and so we have the $*$-subalgebras $D_i \subseteq D$ defined as $D_i = (\mathcal{N}_i \otimes_{\mathrm{alg}} \mathcal{N}_i^{\mathrm{op}}) \rtimes_{\mathrm{alg}} \mathcal{G}$. Note that $\mathcal{N}_1 \subseteq \mathcal{N}_3$, but that the inclusion $\mathcal{N}_2 \subseteq \mathcal{N}_3$ need not hold.

Denote $C = \mathrm{Tr}(p) = \|\Omega\|$. We claim that

$$|\Omega(\Theta(x))| \leqslant C\,\|\Theta_1(x)\| \quad \text{for all } \; x \in D_3 \; . \tag{4.5.5}$$

To prove (4.5.5), first note that in exactly the same way as we proved (4.2.17), we get that (4.5.5) holds for all $x \in D_1$ and thus also for all $x \in D_2$ by norm continuity.

Whenever $(x_n)_n$ and $(a_n)_n$ are sequences in M and A respectively as in the definition of $\mathcal{N}_3$ and $x = \sum_n \Phi(x_n)\rho(a_n)$, we can choose a sequence of projections $p_n \in pMp$ such that $p_n \to p$ in s.o. topology and such that for each fixed n, the series $p_n \sum_i x_i x_i^* p_n$ is norm convergent. This means that for each n, we have that $\Phi(p_n)x \in \mathcal{N}_2$.

Fix $x \in D_3$. Since the automorphisms β_u act as the identity on $\Phi(M)$, it follows that we can find a sequence of projections $p_n \in pMp$ such that $p_n \to p$ in s.o. topology and such that

$$x_n = (\Phi(p_n) \otimes 1 \otimes 1)\, x\, (1 \otimes \Phi(p_n)^{\mathrm{op}} \otimes 1) \in D_2$$

for all n.

Since $\Omega(\pi_\ell(\Phi(x))) = \mathrm{Tr}(pxp)$ for all $x \in M$, we get that

$$\Omega = \Omega \cdot \pi_\ell(p) \quad \text{and} \quad \lim_n \|\Omega - \Omega \cdot \pi_\ell(\Phi(p_n)\| = 0 \; .$$

A similar result holds for π_r and thus,

$$\lim_n \|\Omega - \pi_r(\Phi(p_n)) \cdot \Omega \cdot \pi_\ell(\Phi(p_n))\| = 0 \; .$$

This implies that $\Omega(\Theta(x)) = \lim_n \Omega(\Theta(x_n))$. Since $x_n \in D_2$, we know that

$$|\Omega(\Theta(x_n))| \leqslant C\,\|\Theta_1(x_n)\| \leqslant C\,\|\Theta_1(x)\|$$

for all n. So, (4.5.5) follows.

By (4.5.5), we can define the continuous functional Ω_1 on the C*-algebra $[\Theta_1(D_3)]$ satisfying $\Omega_1(\Theta_1(x)) = \Omega(\Theta(x))$ for all $x \in D_3$. It follows that $\Omega_1(\Theta_1(x)^*\Theta_1(x)) \geqslant 0$ for all $x \in D_3$ and thus, Ω_1 is positive. Extend Ω_1 to a bounded functional on $B(\mathcal{H} \otimes L^2(G))$ without increasing its norm. In particular, Ω_1 remains a positive functional.

Let $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$ be a partial isometry such that $s = v^*v$ and $t = vv^*$ belong to $\mathcal{P}_0$. Using the same notation as in step 4, we define the operator $Y(v) \in B(\mathcal{H})$ given by

$$Y(v) = \pi_r(T(s)^*)\, \pi_r(\Phi(v))\, W_v^*\, \pi_r(T(t)) \; .$$

Note that $Y(v)$ commutes with $\pi_\ell(\Phi(M))$. Also note that $Y(v)^* = Y(v^*)$. Take $u \in \mathcal{N}_{eMe}(A)$ such that $v = us$. Since $W_v = W_u \, \pi_\ell(\rho(s)) \, \pi_r(\rho(s))$, we define the element $y(v) \in D_3$ given by

$$y(v) = (\rho(s) \otimes (\rho(s)\Phi(v)T(s)^*)^{\mathrm{op}} \otimes 1) \, (1 \otimes 1 \otimes u^*) \, (1 \otimes T(t)^{\mathrm{op}} \otimes 1)$$

and note that $\Theta(y(v)) = Y(v)$ and $\Theta_1(y(v)) = Y(v) \otimes 1$.

For every $T \in B(\mathcal{H})$, write $\|T\|_\Omega = \sqrt{\Omega(T^*T)}$. Similarly define $\|T\|_{\Omega_1} = \sqrt{\Omega_1(T^*T)}$ for all $T \in B(\mathcal{H} \otimes L^2(G))$. Applying (4.5.4) for v and v^*, and using that $Y(v^*) = Y(v)^*$, we find that

$$\|\Theta(\Phi(v) \otimes 1 \otimes 1 - y(v))\|_\Omega = \|\pi_\ell(\Phi(v)) - Y(v)\|_\Omega = 0 \text{ and}$$

$$\|\Theta(\Phi(v^*) \otimes 1 \otimes 1 - y(v)^*)\|_\Omega = \|\pi_\ell(\Phi(v^*)) - Y(v)^*\|_\Omega = 0 \; .$$

Then also

$$\begin{aligned} \|(\pi_\ell \circ \Phi \otimes \mathrm{id})(\Phi(v)) - Y(v) \otimes 1\|_{\Omega_1} &= \|\Theta_1(\Phi(v) \otimes 1 \otimes 1 - y(v))\|_{\Omega_1} \\ &= \|\Theta(\Phi(v) \otimes 1 \otimes 1 - y(v))\|_\Omega = 0 \; , \\ \|(\pi_\ell \circ \Phi \otimes \mathrm{id})(\Phi(v^*)) - Y(v)^* \otimes 1\|_{\Omega_1} &= \|\Theta_1(\Phi(v^*) \otimes 1 \otimes 1 - y(v)^*)\|_{\Omega_1} \\ &= \|\Theta(\Phi(v^*) \otimes 1 \otimes 1 - y(v)^*)\|_\Omega = 0 \; . \end{aligned} \tag{4.5.6}$$

Define the positive functional Ω_2 on $q(M \mathbin{\overline{\otimes}} B(L^2(G)))q$ given by $\Omega_2 = \Omega_1 \circ (\pi_\ell \circ \Phi \otimes \mathrm{id})$. Since $Y(v) \otimes 1$ commutes with $\pi_\ell(\Phi(M)) \mathbin{\overline{\otimes}} B(L^2(G))$, it follows from (4.5.6) that $\Omega_2(\Phi(v)T) = \Omega_2(T\Phi(v))$ for every partial isometry $v \in \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)$ with v^*v and vv^* belonging to $\mathcal{P}_0$. We also have that $\Omega_2(\Phi(x)) = \mathrm{Tr}(x)$ for all $x \in pMp$. Since the linear span of all such partial isometries v is $\| \cdot \|_2$-dense in $P = \mathcal{N}^{\mathrm{s}}_{pMp}(pAp)''$, it follows that Ω_2 is $\Phi(P)$-central. So we have proved that P is Φ-amenable. This concludes the proof of Theorem 4.5.1. □

4.6 Strong solidity

In this section, we prove strong solidity and stable strong solidity results for group von Neumann algebras (Theorem H). We do this by applying the dichotomy Theorems 4.2.2 and 4.5.1 to $L(G)$ with the comultiplication $\Delta : L(G) \to L(G) \mathbin{\overline{\otimes}} L(G)$ from (4.1.1) as the coaction Φ.

The following is an easy locally compact analogue of Ozawa's solidity theorem [Oza04].

Proposition 4.6.1. *Let G be a locally compact group with property (S) such that $C_r^*(G)$ is exact. Then, $M = L(G)$ is solid.*

Proof. By the type III version of Ozawa's theorem [Oza04, Theorem 6], as proved in [VV07, Theorem 2.5], it suffices to prove the Akemann-Ostrand property, meaning that the $*$-homomorphism

$$\theta : C_r^*(G) \otimes_{\text{alg}} C_r^*(G) \to \frac{B(L^2(G))}{K(L^2(G))} : \theta(a \otimes b) = \lambda(a)\rho(b) + K(L^2(G))$$

is continuous on the spatial tensor product $C_r^*(G) \otimes_{\min} C_r^*(G)$.

As in the proof of (4.2.9), taking $\zeta : G \to L^2(G)$ as in Proposition 3.1.2 (iii), we define an isometry $Z_0 : L^2(G) \to L^2(G \times G)$ by

$$(Z_0\xi)(s,t) = \zeta(s)(t)\xi(s)$$

for all $\xi \in L^2(G)$ and a.e. $s,t \in G$. We claim that for all $x \in \mathrm{C}_r^*(G)$ the operators

$$Z_0 x - \Delta(x) Z_0 \qquad \text{and} \qquad Z_0\rho(x) - \big(\rho(x) \otimes 1\big) Z_0 \tag{4.6.1}$$

are compact, where again Δ denotes the comultiplication on $L(G)$ given by $\Delta(u_g) = u_g \otimes u_g$ and $\lambda, \rho : \mathrm{C}_r^*(G) \to B\big(L^2(G)\big)$ denote the representations induced by the left and the right regular representations. Indeed, since $C_c(G)$ is $\|.\|$-dense in $\mathrm{C}_r^*(G)$, it suffices to prove (4.6.1) for all $x = \lambda(f)$ with $f \in C_c(G)$. So, fix $f \in C_c(G)$ and denote $T = Z_0\lambda(f) - \Delta\big(\lambda(f)\big)Z_0$. We have

$$\begin{aligned}(T\xi)(s,t) &= \int_G f(u)\big(\zeta(s)(t) - \zeta(u^{-1}s)(u^{-1}t)\big)\xi(u^{-1}s)\,\mathrm{d}u \\ &= \int_G f(su^{-1})\delta_G(u)^{-1}\big(\zeta(s)(t) - \zeta(u)(us^{-1}t)\big)\xi(u)\,\mathrm{d}u \\ &= \int_G k(s,t,u)\xi(u)\,\mathrm{d}u,\end{aligned}$$

for all $\xi \in L^2(G)$ and a.e. $s,t \in G$, where

$$k(s,t,u) = f(su^{-1})\delta_G(u)^{-1}\big(\zeta(s)(t) - \zeta(u)(us^{-1}t)\big).$$

Let $(K_n)_n$ be an increasing sequence of compact subsets of G such that $G = \bigcup_n K_n$. Set $k_n(s,t,u) = 1_{K_n}(s)k(s,t,u)$ and define the operators $T_n : L^2(G) \to L^2(G \times G)$ by

$$(T_n\xi)(s,t) = \int_G k_n(s,t,u)\xi(u)\,\mathrm{d}u.$$

Since f is compactly supported and $\zeta(s) \in L^2(G)$ for all $s \in G$, we have that $k_n \in L^2(G \times G \times G)$ and hence that T_n is a compact operator. Moreover, $T_n \to T$ in norm. Indeed, denoting $L = \operatorname{supp} f$, we have

$$\begin{aligned}
\|(T_n - T)\xi\|_2^2 &= \int_{G\setminus K_n} \int_G \left| f(u)\big(\zeta(s)(t) - \zeta(u^{-1}s)(u^{-1}t)\big)\xi(u^{-1}s)\,\mathrm{d}u\right|^2 \mathrm{d}t\,\mathrm{d}s \\
&\leqslant \int_{G/K_n} \sup_{u\in L} \left\|\zeta(s) - \lambda_u\zeta(u^{-1}s)\right\|_2^2 \left(\int_G |f(u)||\xi(u^{-1}s)|\,\mathrm{d}u\right)^2 \\
&\leqslant \sup_{s\in G\setminus K_n} \sup_{u\in L} \left\|\zeta(s) - \lambda_u\zeta(u^{-1}s)\right\|_2^2 \ \|\lambda(|f||\xi|)\|_2^2 \\
&\leqslant \sup_{s\in G\setminus K_n} \sup_{u\in L} \left\|\zeta(s) - \lambda_u\zeta(u^{-1}s)\right\|_2^2 \ \|f\|_1^2\,\|\xi\|_2^2\,.
\end{aligned}$$

for all $\xi \in L^2(G)$ and $n \in \mathbb{N}$. The convergence now follows from

$$\limsup_{s\to\infty} \left\|\zeta(s) - \lambda_u\zeta(u^{-1}s)\right\|_2^2 = 0$$

uniformly on compact sets for $u \in G$.

Similarly, for $S = Z_0\rho(f) - \big(\rho(f)\otimes 1\big)Z_0$, we have

$$\begin{aligned}
(S\xi)(s,t) &= \int_G f(u)\delta_G(u)^{1/2}\big(\zeta(s)(t) - \zeta(su)(t)\big)\xi(su)\,\mathrm{d}u \\
&= \int_G f(s^{-1}u)\delta_G(s^{-1}u)^{1/2}\big(\zeta(s)(t) - \zeta(u)(t)\big)\xi(u)\,\mathrm{d}u \\
&= \int_G \tilde{k}(s,t,u)\xi(u)\,\mathrm{d}u,
\end{aligned}$$

where

$$\tilde{k}(s,t,u) = f(s^{-1}u)\delta_G(s^{-1}u)^{1/2}\big(\zeta(s)(t) - \zeta(u)(t)\big).$$

Set $\tilde{k}_n(s,t,u) = 1_{K_n}(s)k(s,t,u)$, and define compact operators $S_n : L^2(G) \to L^2(G\times G)$ by

$$(S_n\xi)(s) = \int_G \tilde{k}_n k(s,t,u)\xi(u)\,\mathrm{d}u.$$

Using that

$$\limsup_{s\to\infty} \|\zeta(s) - \zeta(su)\|_2^2 = 0$$

uniformly on compact sets for $u \in G$, a similar calculation as above yields that $S_n \to S$ in norm. This proves (4.6.1).

Now, by (4.6.1),

$$\theta(x \otimes y) = Z_0^* Z_0 \lambda(x)\rho(y) + K\big(L^2(G)\big) = Z_0^* \Delta(x)\big(\rho(x) \otimes 1\big) Z_0 + K\big(L^2(G)\big).$$

Defining the unitary $U : L^2(G \times G) \to L^2(G \times G)$ by

$$(U\xi)(s,t) = \xi(s, t^{-1}s)\delta_G(t^{-1}s)^{1/2}$$

for all $\xi \in L^2(G \times G)$ and a.e. $s, t \in G$, a straightforward calculation yields that

$$U(x \otimes y)U^* = \Delta(x)\big(\rho(y) \otimes 1\big)$$

for all $x, y \in \mathrm{C}_r^*(G)$ and hence

$$\theta(x \otimes y) = Z_0^* U(x \otimes y) U^* Z_0 + K\big(L^2(G)\big)$$

for all $x, y \in \mathrm{C}_r^*(G)$. We conclude that θ is indeed continuous with respect to the spatial tensor norm. □

The following is the first part of Theorem H from the introduction. It proves strong solidity for certain group von Neumann algebras of unimodular groups.

Theorem 4.6.2. *Let G be a unimodular, weakly amenable, locally compact group in class $\mathcal{S}$ and assume that $L(G)$ is diffuse. Then, for every finite trace projection $p \in L(G)$, we have that $pL(G)p$ is strongly solid.*

Proof. Fix a Haar measure on G and denote by Tr the associated faithful, normal, semifinite trace on $M = L(G)$. Fix a projection $p \in L(G)$ with $\mathrm{Tr}(p) < \infty$. We have to prove that pMp is strongly solid. So, fixing a diffuse amenable von Neumann subalgebra $A \subseteq pMp$, we have to prove that $\mathcal{N}_{pMp}(A)''$ is amenable.

Let $\Delta : L(G) \to L(G)$ be the comultiplication from (4.1.1). Viewing Δ as a coaction on M, we can apply Theorem 4.2.2. Since A is amenable, we certainly have that A is Δ-amenable. We prove that A cannot be Δ-embedded.

Since A is diffuse, we can take a net $a_n \in \mathcal{U}(A)$ such that $a_n \to 0$ weakly. For every $\xi, \eta \in L^2(G)$, denote by $\omega_{\xi,\eta} \in L(G)_*$ the vector functional given by $\omega_{\xi,\eta}(\lambda_g) = \langle \lambda_g \xi, \eta \rangle$. Also denote by $m_{\xi,\eta} = (\mathrm{id} \otimes \omega_{\xi,\eta}) \circ \Delta$ the associated normal completely bounded map. We claim that

$$m_{\xi,\eta}(a_n) \to 0 \quad \text{in s.o. topology} \tag{4.6.2}$$

for all $\xi, \eta \in L^2(G)$. Fix $\xi, \eta \in L^2(G)$ and fix $\mu \in L^2(G)$. To prove (4.6.2), we have to prove that $\|m_{\xi,\eta}(a_n)\mu\| \to 0$. Denote by $U \in L^\infty(G) \mathbin{\overline{\otimes}} L(G)$ the unitary from (4.1.1) satisfying $U(x \otimes 1)U^*$. Using Lemma 4.2.4, we have for every $a \in L(G)$ that

$$m_{\xi,\eta}(a)\mu = (\mathrm{id} \otimes \omega_{\xi,\eta})(\Delta(a))\mu = (\omega_{\xi,\eta} \otimes \mathrm{id})(\Delta(a))\mu$$
$$= (\omega_{\xi,\eta} \otimes \mathrm{id})(U(a \otimes 1)U^*)\mu = (\eta^* \otimes 1)U(a \otimes 1)U^*(\xi \otimes \mu) \; .$$

Approximating $U^*(\xi \otimes \mu) \in L^2(G) \otimes L^2(G)$ by linear combinations of vectors $\xi_0 \otimes \mu_0$, it suffices to prove that

$$\lim_n \|(\eta^* \otimes 1)U(a_n \otimes 1)(\xi_0 \otimes \mu_0)\| = 0$$

for all $\eta, \xi_0, \mu_0 \in L^2(G)$. Since

$$\big((\eta^* \otimes 1)U(1 \otimes \mu_0)\xi\big)(h) = \int_G \mu_0(g^{-1}h)\overline{\eta(g)}\xi(g)\, \mathrm{d}g,$$

we see that $T = (\eta^* \otimes 1)V(1 \otimes \mu_0)$ is an integral operator with square integrable kernel and hence compact. It follows that

$$\lim_n \|(\eta^* \otimes 1)U(a_n \otimes 1)(\xi_0 \otimes \mu_0)\| = \lim_n \|Ta_n\xi_0\| = 0$$

We conclude that the claim in (4.6.2) is proved.

Let $\mathcal{H} = \Delta(p)\big(L^2(M) \otimes L^2(G)\big)$ be the pMp-M bimodule defined by $x \cdot \xi \cdot y = \Delta(x)\xi(y \otimes 1)$ for $x \in pMp$, $y \in M$ and $\xi \in \mathcal{H}$ from Definition 4.2.1. Using that $L^2(M) = L^2(G)$, note that for all $\mu_1, \mu_2 \in L^2(M)$ and for all $\xi, \eta \in L^2(G)$, we have

$$\langle \Delta(a_n)\,(\mu_1 \otimes \xi)\,a_n^*, \mu_2 \otimes \eta\rangle = \langle m_{\xi,\eta}(a_n)\mu_1, \mu_2 a_n\rangle \; .$$

Together with (4.6.2) this implies that

$$\langle \Delta(a_n) \cdot \xi \cdot a_n^*, \xi\rangle = 0$$

for all $\xi \in \mathcal{H}$. So, there is no nonzero A-central vector in $\mathcal{H}$, meaning that A cannot be Δ-embedded.

Write $P = \mathcal{N}_{pMp}(A)''$. By Theorem 4.2.2, P is Δ-amenable, meaning that $\mathcal{H}$ is left P-amenable. Define the unitary $V : L^2(G) \otimes L^2(G) \to L^(G) \otimes L^2(G)$ by

$$(V\xi)(s,t) = \xi(t, t^{-1}s)$$

for $\xi \in L^2(G) \otimes L^2(G)$ and a.e. $s, t \in G$. Note that $V(x \otimes 1)V^* = \Delta(x)$ and $V(JxJ \otimes 1)V^* = 1 \otimes JxJ$ for all $x \in M$. Hence, V yields a unitary equivalence between the bimodule $\mathcal{H}$ to the pMp-M-bimodule $\mathcal{K} = L^2(pM) \otimes L^2(G)$ given by $x(\xi \otimes \eta)y = x\xi \otimes \eta y$ for $x \in pMp$, $y \in M$, $\xi \in L^2(pM)$ and $\eta \in L^2(G)$. Hence, $\mathcal{K}$ is left P-amenable which, by definition, yields a conditional expectation $\Phi : B\big(L^2(pM)\big) \overline{\otimes} M \to P$. Restricting Φ to $B\big(L^2(pM)\big) \otimes 1$ yields that $P = \mathcal{N}_{pMp}(A)''$ is amenable. We conclude that pMp is indeed strongly solid. □

If we strengthen the assumption of weak amenability to CMAP, then we can prove the following *stable* strong solidity result for *nonunimodular* groups. It is the second part of Theorem H from the introduction.

Theorem 4.6.3. *Let G be a locally compact group with CMAP and in class $\mathcal{S}$. Assume that $L(G)$ is diffuse and that the kernel of the modular function $G_0 = \{g \in G \mid \delta(g) = 1\}$ is an open subgroup of G. Then, $L(G)$ is stably strongly solid.*

For the proof of this result, we need the following lemma.

Lemma 4.6.4. *Let M be a diffuse von Neumann algebra and $p_n \in M$ a sequence of projections such that $p_n \to 1$ in s.o. topology. Then M is stably strongly solid if and only if $p_n M p_n$ is stably strongly solid for every n.*

Proof. Write $\mathcal{H} = \ell^2(\mathbb{N})$ and denote by $z_n \in \mathcal{Z}(M)$ the central support of p_n. We have $B(\mathcal{H}) \mathbin{\overline{\otimes}} p_n M p_n \cong B(\mathcal{H}) \mathbin{\overline{\otimes}} M z_n$. By [BHV18, Corollary 5.2], we get that $p_n M p_n$ is stably strongly solid if and only if $M z_n$ is stably strongly solid.

We prove that M is stably strongly solid if and only if $M z_n$ is stably strongly solid for each n. Suppose first that that M is stably strongly solid. Take a diffuse, amenable von Neumann subalgebra $A \subseteq M z_n$ with expectation. Since also $M(1 - z_n)$ is diffuse, by [CS78, Corollary 8] we can take a diffuse amenable von Neumann subalgebra $B \subseteq M(1 - z_n)$ with expectation. Now, $C = A + B \subseteq M$ is a diffuse, amenable von Neumann subalgebra with expectation. Since M is stably strongly solid, the stable normalizer $\mathcal{N}^{\mathrm{s}}_M(C)''$ is amenable. But since $A = C z_n$, we have $\mathcal{N}^{\mathrm{s}}_{M z_n}(A) = \mathcal{N}^{\mathrm{s}}_M(C) z$ and hence $\mathcal{N}^{\mathrm{s}}_{M z_n}(A)''$ is amenable.

Conversely, suppose that each $M z_n$ is stably strongly solid. Let $A \subseteq M$ be a diffuse, amenable von Neumann subalgebra with expectation. Denote $P = \mathcal{N}_M(A)''$. Then, each $A z_n \subseteq M z_n$ is a diffuse, amenable von Neumann subalgebra with expectation. Hence, each $P_n = \mathcal{N}^{\mathrm{s}}_{M z_n}(A z_n)'' = P z_n$ is amenable. Let $E_n : B(\mathcal{H} z_n) \to P z_n$ be the corresponding conditional expectations. Then, the weak* limit point of $T \mapsto E_n(z_n T z_n)$ is a conditional expectation $E : B(\mathcal{H}) \to P$, thus proving the amenability of P. □

Proof of Theorem 4.6.3. Let G be a locally compact with CMAP and in class $\mathcal{S}$. Assume that he kernel G_0 of the modular function $\delta_G : G \to \mathbb{R}^+$ is an open subgroup of G. Fix a left Haar weight on G and denote by φ the associated Plancherel weight on $M = L(G)$ (see Section 2.4.4). Recall from (2.4.2) that the modular automorphism group σ^φ is given by

$$\sigma_t^\varphi(u_g) = \delta(g)^{it}\, u_g$$

for all $g \in G$ and $t \in \mathbb{R}$. So, $L(G_0)$ lies in the centralizer $L(G)^\varphi$ and since $G_0 \subseteq G$ is an open subgroup, the restriction of φ to $L(G_0)$ is semifinite. By Lemma 4.6.4, it is sufficient to prove that $pL(G)p$ is stably strongly solid for each nonzero projection $p \in L(G_0)$ with $\varphi(p) < \infty$. Fix such a projection p and let $A \subseteq pMp$ be a diffuse amenable von Neumann subalgebra with expectation. Write $P = \mathcal{N}^{\mathrm{s}}_{pMp}(A)''$. We have to prove that P is amenable.

Denote $\mathcal{H} = \ell^2(\mathbb{N})$ and define $M_1 = B(\mathcal{H}) \overline{\otimes} M$. Write $A_0 = B(\mathcal{H}) \overline{\otimes} A$ and $p_1 = 1 \otimes p$. By Proposition 2.4.69, it suffices to prove that $\mathcal{N}_{p_1 M_1 p_1}(A_0)''$ is amenable. Since G has CMAP, certainly G is exact (see Theorem 2.3.33) and Proposition 4.6.1 implies that $A' \cap pMp$ is amenable. So, $A_1 := A_0 \vee (A_0' \cap p_1 M_1 p_1)$ is amenable. Indeed, by Theorem 2.4.61 both A_0 and $A_0' \cap p_1 M_1 p_1$ are AFD and hence so is A_1. Since $\mathcal{N}_{p_1 M_1 p_1}(A_0)'' \subseteq \mathcal{N}_{p_1 M_1 p_1}(A_1)''$ and since this is an inclusion with expectation (see Proposition 2.4.21 (b)), it suffices to show that $P_1 := \mathcal{N}_{p_1 M_1 p_1}(A_1)''$ is amenable.

Let $e \in B(\mathcal{H})$ be a minimal projection and choose a faithful normal state η on $B(\mathcal{H})$ such that e belongs to the centralizer of η. Also choose a faithful normal state ψ on pMp such that $\sigma_t^\psi(A) = A$ for all $t \in \mathbb{R}$. Note that such a state indeed exists by Theorem 2.4.19. Now, $\eta \otimes \psi$ is a faithful normal state on $p_1 M_1 p_1$ and the subalgebras A_1 and P_1 are globally invariant under $\sigma^{\eta\otimes\psi}$ and we obtain the canonical inclusions of continuous cores

$$\mathrm{c}_{\eta\otimes\psi}(A_1) \subseteq \mathrm{c}_{\eta\otimes\psi}(P_1) \subseteq \mathrm{c}_{\eta\otimes\psi}(p_1 M_1 p_1) .$$

Since $A_1' \cap p_1 M_1 p_1 = \mathcal{Z}(A_1)$, it follows from [BHV18, Lemma 4.1] that $\mathrm{c}_{\eta\otimes\psi}(P_1)$ is contained in the normalizer of $\mathrm{c}_{\eta\otimes\psi}(A_1)$. By Takesaki's duality theorem [Tak03a, Theorem X.2.3], P_1 is amenable if and only if its continuous core is amenable. So, it suffices to prove that the normalizer of $\mathrm{c}_{\eta\otimes\psi}(A_1)$ is amenable. Now we can cut down again with the projection $e \otimes 1$ and conclude that it is sufficient to prove the following result: for any diffuse amenable $B \subseteq pMp$ with expectation and for every faithful normal state ψ on pMp with $\sigma_t^\psi(B) = B$ for all $t \in \mathbb{R}$, the canonical subalgebra $\mathrm{c}_\psi(B)$ of $\mathrm{c}_\psi(pMp)$ has an amenable stable normalizer.

Whenever $p' \in M^\varphi$ is a projection with $p' \geqslant p$ and $\varphi(p') < \infty$, we can realize the continuous core of $p'Mp'$ as $\pi_\varphi(p')\mathcal{M}\pi_\varphi(p')$ where $\mathcal{M} = \mathrm{c}_\varphi(M)$ and $\pi_\varphi : M \to \mathcal{M}$ is the inclusion. Recall that $p \in L(G_0) \subseteq M^\varphi$. Let

$$\Pi : \mathrm{c}_\psi(pMp) \to \pi_\varphi(p)\mathcal{M}\pi_\varphi(p)$$

be the canonical trace preserving isomorphism. Let $p_n \in L_\psi(\mathbb{R})$ be a sequence of projections having finite trace and converging to 1 in s.o. topology. Since B is diffuse, it follows from [HU15, Lemma 2.5] that

$$\Pi(p_n\, \mathrm{c}_\psi(B) p_n) \quad \not\prec_{\pi_\varphi(p')\mathcal{M}\pi_\varphi(p')} \quad L_\varphi(\mathbb{R})\pi_\varphi(p') \tag{4.6.3}$$

whenever $p' \in M^\varphi$ is a projection with $p' \geqslant p$ and $\varphi(p') < \infty$. Denote by $\mathcal{P}$ the set of these projections p' and define the $*$-algebra

$$\mathcal{M}_0 := \bigcup_{p' \in \mathcal{P}} \pi_\varphi(p') \mathcal{M} \pi_\varphi(p') \; .$$

There is a unique linear map $E : \mathcal{M}_0 \to L_\varphi(\mathbb{R})$ such that for every $p' \in \mathcal{P}$, the restriction of E to $\pi_\varphi(p')\mathcal{M}\pi_\varphi(p')$ is normal and given by $E(\pi_\varphi(x)\lambda_\varphi(t)) = \varphi(x)\lambda_\varphi(t)$ for all $x \in p'Mp'$ and $t \in \mathbb{R}$. Note that this restriction of E can be viewed as $\varphi(p')$ times the unique trace preserving conditional expectation of $\pi_\varphi(p')\mathcal{M}\pi_\varphi(p')$ onto $L_\varphi(\mathbb{R})p'$.

Combining (4.6.3) and with Theorem 2.5.33, in order to prove that $\mathrm{c}_\psi(B)$ has an amenable stable normalizer inside $\mathrm{c}_\psi(pMp)$, it is sufficient to prove the following statement: whenever $q \in \pi_\varphi(p)\mathcal{M}\pi_\varphi(p)$ is a projection of finite trace and $A \subseteq q\mathcal{M}q$ is a von Neumann subalgebra that admits a net of unitaries $v_n \in \mathcal{U}(A)$ satisfying

$$E(x^* v_n y) \to 0 \quad \text{in s.o. topology,} \tag{4.6.4}$$

for all $x, y \in \mathcal{M}_0$, then the stable normalizer of A inside $q\mathcal{M}q$ is amenable. Fix such a von Neumann subalgebra $A \subseteq q\mathcal{M}q$ and fix a net of unitaries $v_n \in \mathcal{U}(A)$ satisfying (4.6.4).

Since $\Delta \circ \sigma_t^\varphi = (\sigma_t^\varphi \otimes \mathrm{id}) \circ \Delta$ for all $t \in \mathbb{R}$, there is a well defined coaction given by

$$\Phi : \mathcal{M} \to \mathcal{M} \mathbin{\overline{\otimes}} L(G) : \Phi(\pi_\varphi(x)\lambda_\varphi(t)) = (\pi_\varphi \otimes \mathrm{id})(\Delta(x))\,(\lambda_\varphi(t) \otimes 1)$$

for all $x \in M$, $t \in \mathbb{R}$.

The $\mathcal{M}$-bimodule ${}_{\Phi(\mathcal{M})}(L^2(\mathcal{M}) \otimes L^2(G))_{\mathcal{M} \otimes 1}$ is isomorphic to $L^2(\mathcal{M}) \otimes_{L_\varphi(\mathbb{R})} L^2(\mathcal{M})$ and thus weakly contained in the coarse $\mathcal{M}$-bimodule. Writing $P = \mathcal{N}^{\mathrm{s}}_M(A)''$, the left P-amenability of ${}_{\Phi(\mathcal{M})}\Phi(p)(L^2(\mathcal{M}) \otimes L^2(G))_{\mathcal{M} \otimes 1}$ implies left P-amenability of the coarse $p\mathcal{M}p$-$\mathcal{M}$-bimodule $L^2(p\mathcal{M}) \otimes L^2(\mathcal{M})$ and hence amenability of P. Using Theorem 4.5.1, it only remains to prove that (4.6.4) implies that A cannot be Φ-embedded.

We deduce that A cannot be Φ-embedded from the following approximation result: for all $a \in C_r^*(G_0)$, $\omega \in L(G)_*^+$ and $\varepsilon > 0$, there exist $n \in \mathbb{N}$, elements $a_j, x_j \in L(G)$ and scalars $\delta_j > 0$ for $j \in \{1, \dots, n\}$ such that

$$\sigma_t^\varphi(a_j) = \delta_j^{it} a_j \quad \text{and} \quad \sigma_t^\varphi(x_j) = \delta_j^{-it} x_j \tag{4.6.5}$$

for all $j \in \{1, \dots, n\}$ and all $t \in \mathbb{R}$, and such that the map

$$\Psi : \mathcal{M} \to \mathcal{M} : x \mapsto \sum_{j,k=1}^{n} \pi_\varphi(x_j^*)\, E\big(\pi_\varphi(pa_j)^* x \pi_\varphi(pa_k)\big)\, \pi_\varphi(x_k) \tag{4.6.6}$$

is normal and completely bounded, and satisfies

$$\big\|(\mathrm{id}\otimes\omega)\Phi\big(\pi_\varphi(pa)^*\,\cdot\,\pi_\varphi(pa)\big)-\Psi\big\|_{\mathrm{cb}}<\varepsilon\,. \tag{4.6.7}$$

Already note that (4.6.5) implies that the right support of pa_j belongs to $\mathcal{P}$, so that $\pi_\varphi(pa_j)\in\mathcal{M}_0$ and the map Ψ is well defined and normal.

Assuming that such an approximation exists, we already deduce that A cannot be Φ-embedded. Indeed, considering the sequence of unitaries $(v_n)_n$ from (4.6.4), it suffices to prove that

$$\lim_n\langle\Phi(v_n)(\mu\otimes\xi),\mu v_n\otimes\xi\rangle=0\quad\text{for all }\mu\in L^2(\mathcal{M})\text{ and }\xi\in L^2(G),$$

because then also $\lim_n\langle\Phi(v_n)\eta(v_n^*\otimes 1),\eta\rangle=0$ for all $\eta\in\Delta(q)L^2(\mathcal{M})\otimes L^2(G)$, excluding the existence of a nonzero vector $\eta\in\Delta(q)L^2(\mathcal{M})\otimes L^2(G)$ satisfying $\Phi(a)\eta=\eta(a\otimes 1)$ for all $a\in A$.

Since $\mu\otimes\xi$ can be approximated by vectors of the form $\Phi(\pi_\varphi(a))(\mu\otimes\xi)$ with $a\in \mathrm{C}^*_r(G_0)$, it suffices to prove that

$$\lim_n\langle\Phi(\pi_\varphi(a)^*v_n\pi_\varphi(a))(\mu\otimes\xi),\mu v_n\otimes\xi\rangle=0$$

for all $a\in \mathrm{C}^*_r(G_0)$, $\mu\in L^2(\mathcal{M})$ and $\xi\in L^2(G)$. Since every $\xi\in L^2(G)$ implements a positive normal functional $\omega_\xi\in L(G)^+_*$, it suffices to prove that

$$(\mathrm{id}\otimes\omega)\big(\Phi(\pi_\varphi(a)^*v_n\pi_\varphi(a))\big)\to 0$$

in s.o. topology for all $a\in C^*_r(G_0)$ and $\omega\in L(G)^+_*$. But this follows by the approximation in (4.6.7) and because (4.6.4) implies that $\Psi(v_n)\to 0$ in s.o. topology for every Ψ of the form (4.6.6).

Fixing $a\in C^*_r(G_0)$, $\omega\in L(G)^+_*$ and $\varepsilon>0$, it remains to find the approximation as in (4.6.7).

First take $\xi\in C_c(G)$ such that the vector functional ω_ξ satisfies $\|\omega-\omega_\xi\|<(1/3)\,\varepsilon\,\|a\|^{-2}$. It follows that, as maps on $M=L(G)$,

$$\big\|(\mathrm{id}\otimes\omega)\Delta\big(a^*p\,\cdot\,pa\big)-(\mathrm{id}\otimes\omega_\xi)\Delta\big(a^*p\,\cdot\,pa\big)\big\|_{\mathrm{cb}}<\frac{\varepsilon}{3}\,. \tag{4.6.8}$$

Fix $F\in C_c(G)$ with $0\leqslant F\leqslant 1$ and $\xi=F\xi$. Viewing F as a multiplication operator in $L^\infty(G)$ and using the unitary $U\in L^\infty(G)\mathbin{\overline{\otimes}}L(G)$ satisfying $U(x\otimes 1)U^*=\Delta(x)$ from (4.1.1), we have for every $x\in M$

$$\begin{aligned}(\mathrm{id}\otimes\omega_\xi)\big(\Delta\big(a^*p\,x\,pa\big)\big)&=(\omega_\xi\otimes\mathrm{id})\big(\Delta\big(a^*p\,x\,pa\big)\big)\\&=(\xi^*\otimes 1)U(a^*pxpa\otimes 1)U^*(\xi\otimes 1)\end{aligned}$$

$$= (\xi^* \otimes 1)U(Fa^* \, pxp \, aF \otimes 1)U^*(\xi \otimes 1) \; .$$

Since $a \in \mathrm{C}^*_r(G_0) \subseteq \mathrm{C}^*_r(G)$ and $F \in C_c(G)$, we get that aF is a compact operator on $L^2(G)$ that commutes with the modular operator Δ_G associated to the GNS-representation of $L(G)$ on $L^2(G)$. Indeed, take $a_n \in C_c(G)$ such that $\lambda(a_n) \to a$ in norm. Then $a_n F \to aF$ in norm and for every $n \in \mathbb{N}$ and every $\xi \in L^2(G)$, we have

$$(a_n F\xi)(t) = \int_G a_n(s)F(s^{-1}t)\xi(s^{-1}t)\,\mathrm{d}s = \int_G a_n(ts^{-1})\delta_G(s)^{-1}F(s)\xi(s)\,\mathrm{d}s.$$

So, each $a_n F$ is an integral operator with square integrable kernel and hence compact.

We can approximate aF be a finite rank operator T of the form

$$T = \sum_{j=1}^{n} \mu_j \xi_j^*$$

where $\xi_j, \mu_j \in C_c(G) \subseteq L^2(G)$ and $\Delta_G \xi_j = \delta_j \xi_j$, $\Delta_G \mu_j = \delta_j \mu_j$, and such that $\|T\| \leqslant \|aF\| \leqslant \|a\|$ and

$$\|aF - T\| \leqslant \frac{\varepsilon}{3\,\|a\|\,\|\xi\|^2} \; .$$

Defining

$$m : M \to M : m(x) = (\mathrm{id} \otimes \omega)(\Delta(a^* \, pxp \, a)) \;\; ,$$

$$m_1 : M \to M : m_1(x) = (\xi^* \otimes 1)V(T^* \, pxp \, T \otimes 1)V^*(\xi \otimes 1) \; ,$$

we get that $\|m - m_1\|_{\mathrm{cb}} < \varepsilon$.

Defining

$$a_j = \lambda(\mu_j) = \int_G \mu_j(g)\,\lambda_g \, dg \quad \text{and} \quad x_j = (\omega_{\xi,\xi_j} \otimes \mathrm{id})(V^*) \; ,$$

we get that

$$m_1(x) = \sum_{j,k=1}^{n} \varphi(a_j^* \, pxp \, a_k) x_j^* x_k \; .$$

Both m and m_1 commute with the modular automorphism group σ^φ and thus canonically extend to $\mathcal{M} = \mathrm{c}_\varphi(M)$ by acting as the identity on $L_\varphi(\mathbb{R})$. The canonical extension of m equals

$$(\mathrm{id} \otimes \omega)\Phi\big(\pi_\varphi(pa)^* \; \cdot \; \pi_\varphi(pa)\big) \; ,$$

while the canonical extension of m_1 equals the map Ψ given by (4.6.6). Since $\|m - m_1\|_{\mathrm{cb}} < \varepsilon$, also (4.6.7) holds and the theorem is proved. □

4.7 Unique prime factorization

Proposition 4.6.1 implies that $L(G)$ is a prime factor whenever G is a locally compact group in class $\mathcal{S}$ whose group von Neumann algebra is a nonamenable factor. In this section, we prove Theorem I, i.e. we prove that tensor products of such factors have unique prime factorization.

Theorem 4.7.1. *Let $G = G_1 \times \cdots \times G_m$ be a direct product of locally compact groups in class $\mathcal{S}$ whose group von Neumann algebras $L(G_i)$ are nonamenable factors. If*

$$L(G) \cong N_1 \mathbin{\overline{\otimes}} \ldots \mathbin{\overline{\otimes}} N_n$$

for some factors N_i that are not of type I, then $n \leqslant m$. Moreover, all factors N_i are prime if and only if $n = m$ and in that case (after relabeling) $L(G_i)$ is stably isomorphic to N_i for $i = 1, \ldots, n$.

To prove this theorem, we use the following property introduced by Houdayer and Isono [HI17].

Definition 4.7.2. Let $(M, \mathcal{H}, J, P)$ be a von Neumann algebra in standard form. We say that M satisfies the *strong condition (AO)* if there exist C*-algebras $A \subseteq M$ and $\mathcal{C} \subseteq B(\mathcal{H})$ such that

- A is exact and w.o. dense in M,
- $\mathcal{C}$ is nuclear and contains A,
- all commutators $[c, JaJ]$ for $c \in \mathcal{C}$ and $a \in A$ belong to the compact operators $K(\mathcal{H})$.

Note that the definition in [HI17, Definition 2.6] also requires A and $\mathcal{C}$ to be unital. However, by Proposition 2.3.22 this requirement is not essential. The *class (AO)* is now defined as the smallest class of von Neumann algebras containing all (separable) von Neumann algebras satisfying strong condition (AO) and that is stable under taking von Neumann subalgebras with expectation.

In [HI17, Theorems B], Houdayer and Isono provide a unique factorization result for nonamenable factors in class (AO). A slightly more general version, not requiring the factors N_i in the formulation below to have a state with large centralizer, was later proved by Ando, Haagerup, Houdayer, and Marrakchi in [AHHM18, Application 4]. This more general version is as follows.

Theorem 4.7.3. *Let $M \cong M_1 \mathbin{\overline{\otimes}} \ldots \mathbin{\overline{\otimes}} M_m$ be a tensor product of nonamenable factors in class (AO). If $M \cong N_1 \mathbin{\overline{\otimes}} \ldots \mathbin{\overline{\otimes}} N_n$ for some factors N_i that are not* *of type I, then $n \leqslant m$. Moreover, all factors N_i are prime if and only if $n = m$ and in that case (after relabeling) M_i is stably isomorphic to N_i for $i = 1, \ldots, n$.*

Theorem 4.7.1 now follows immediately by combining Theorem 4.7.3 with the following result.

Proposition 4.7.4. *Let G be a locally compact group in class $\mathcal{S}$. Then, its group von Neumann algebra $L(G)$ satisfies strong condition (AO).*

Proof. Recall that $L(G)$ is in standard form on $L^2(G)$ and that for any $f \in C_c(G)$, we have

$$\big(J\lambda(f)J\xi\big)(s) = (\rho(f)\xi)(s) = \int_G \overline{f(t)}\delta_G(t)^{1/2}\xi(st)\,\mathrm{d}t.$$

Let $A = \mathrm{C}^*_r(G)$ be the reduced group C^*-algebra of G. Then, obviously A is exact and w.o. dense in $L(G)$. Recall from Theorem 3.2.3 that the action $G \curvearrowright h^uG$ is topologically amenable and hence the crossed product $C(h^uG) \rtimes G$ is nuclear (see Theorem 2.3.30). Now, the inclusion $C(h^uG) \subseteq C_b^u(G) \hookrightarrow B(L^2(G))$ together with the unitary representation $g \mapsto \lambda_g$ induces a $*$-morphism $\pi : C(h^uG) \rtimes G \to B(L^2(G))$. The image $\mathcal{C} = \pi\big(C(h^uG) \rtimes G\big)$ is nuclear as a quotient of a nuclear C^*-algebra (see Theorem 2.3.25), and obviously contains A. The algebra $C_c\big(G, C(h^uG)\big)$ is a dense subalgebra in $C(h^uG) \rtimes G$. Identifying an element $h \in C_c\big(G, C(h^uG)\big) \subseteq C(h^uG) \rtimes G$ with a function on $G \times G$ that is compactly supported in the first component, we get that the action $\pi(h)$ on a $\xi \in L^2(G)$ is given by

$$\big(\pi(h)\xi\big)(s) = \int_G h(t,s)\xi(t^{-1}s)\,\mathrm{d}t$$

for $h \in C_c(G, C(h^uG))$, $\xi \in L^2(G)$ and a.e. $s \in G$. Denote by $\mathcal{C}_0$ the image of $C_c(G, C(h^uG))$ under π.

We prove that $\mathcal{C}$ commutes with JAJ up to the compact operators. Since $C_c(G)$ is dense in $\mathrm{C}^*_r(G)$ and $\mathcal{C}_0$ is dense in $\mathcal{C}$, it suffices to prove that for every $f \in C_c(G)$ and every $\pi(h) \in \mathcal{C}_0$, we have $T = [\pi(h), J\lambda(f)J] \in K(L^2(G))$. A straightforward calculation yields that for $\xi \in L^2(G)$ and $s \in G$, we have

$$\begin{aligned}(T\xi)(s) &= \int_G\int_G \big(h(t,s) - h(t,su)\big)\overline{f(u)}\delta_G(u)^{1/2}\xi(t^{-1}su)\,\mathrm{d}t\,\mathrm{d}u \\ &= \int_G\int_G \big(h(t,s) - h(t,tu)\big)\overline{f(s^{-1}tu)}\delta_G(s^{-1}tu)^{1/2}\,\mathrm{d}t\,\mathrm{d}u \\ &= \int_G k(s,u)\xi(u)\,\mathrm{d}u,\end{aligned}$$

where

$$k(s,u) = \int_G \big(h(t,s) - h(t,tu)\big)\overline{f(s^{-1}tu)}\delta_G(s^{-1}tu)^{1/2}\,\mathrm{d}t.$$

Let $(K_n)_n$ be an increasing sequence of compact subsets of G such that $G = \bigcup_n K_n$. Set $k_n(s,u) = 1_{K_n}(s)k(s,u)$ and define the operator $T_n \in B(L^2(G))$ by

$$(T_n\xi)(s) = \int_G k(s,u)\xi(u)\,\mathrm{d}u$$

Note that since $f \in C_c(G)$ and h is compactly supported in the first component, we have that each $k_n \in L^2(G \times G)$ and hence that T_n is compact. Moreover, $T_n \to T$ in norm. Indeed, if $L \subseteq G$ is a compact subset containing the support of f of (the first component of) h, then

$$\begin{aligned}
&\|T\xi - T_n\xi\|^2 \\
&\quad = \int_{G\setminus K_n} \left| \int_G \int_G \big(h(t,s) - h(t,su)\big)\overline{f(u)}\delta_G(u)^{1/2}\xi(t^{-1}su)\,\mathrm{d}t\,\mathrm{d}u \right|^2 \mathrm{d}s \\
&\quad \leqslant \int_{G\setminus K_n} \sup_{t,u\in L} |h(t,s) - h(t,su)|^2 \left(\int_L \int_L |f(u)|\ \delta_G(u)^{1/2}\ |\xi(t^{-1}su)|\,\mathrm{d}u\,\mathrm{d}t \right)^2 \mathrm{d}s \\
&\quad \leqslant \sup_{s\in G\setminus K_n} \sup_{t,u\in L} |h(t,s) - h(t,su)|^2\ \mu(L)^2\ \|J\lambda(|f|)J|\xi|\|_2^2 \\
&\quad = \sup_{s\in G\setminus K_n} \sup_{t,u\in L} |h(t,s) - h(t,su)|^2\ \mu(L)^2\ \|f\|_1^2\ \|\xi\|_2^2
\end{aligned}$$

and

$$\limsup_{s\to\infty} |h(t,s) - h(t,su)|^2 = 0$$

uniformly on compact sets for $t, u \in G$. We conclude that T itself is compact. $\square$

Chapter 5

Conclusion

In this thesis, we proved the first W^*-rigidity results for group von Neumann algebras and group measure space von Neumann algebras of locally compact groups. In order to do this we defined Ozawa's class $\mathcal{S}$ (Definition A) for locally compact groups and provided examples of groups in this class (Proposition C and Theorem D). We also proved that class $\mathcal{S}$ is closed under measure equivalence (Theorem E) and we provided a characterization of group in class $\mathcal{S}$ (Theorem B), similar to Ozawa's result [Oza06, Theorem 4.1] for discrete groups.

We were then able to prove that the canonical Cartan subalgebra in crossed products $L^\infty(X) \rtimes G$ is unique, whenever G is a direct product of nonamenable, weakly amenable, locally compact groups in class $\mathcal{S}$ and $G \curvearrowright (X, \mu)$ is an arbitrary free, pmp action (see Theorem F, or Theorem 4.3.1 for a more general version). We deduced this result from a very general dichotomy type result on the normalizer $\mathcal{N}_M(A)'' = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ of a subalgebra A in a von Neumann algebra M equipped with a coaction (Theorem 4.2.2). The proof of this result followed the same lines as the earlier results [PV14a, Theorem 5.1] and [PV14b, Theorem 3.1], but in a much more abstract setting. In Theorem 4.5.1, we were able to obtain a similar result on the stable normalizer $\mathcal{N}^{\mathrm{s}}_M(A) = \{x \in M \mid xAx^* \subseteq A \text{ and } x^*Ax \subseteq A\}$ by adapting the methods of [BHV18] to this abstract setting.

Using the uniqueness theorem of Cartan subalgebras we obtained a W^*-strong rigidity result for product groups (Theorem G). We obtained this result by first generalizing the cocycle superrigidity theorem of [MS04] and the orbit equivalence rigidity theorem of [Sak09b] to the locally compact setting.

Using the dichotomy results Theorems 4.2.2 and 4.5.1, we obtained strong solidity results on $L(G)$ when G is unimodular, weakly amenable groups in class $\mathcal{S}$ and stable strong solidity results when G is possibly nonunimodular, but has CMAP and is in class $\mathcal{S}$ (Theorem H). Using the characterization in Theorem B, we obtained unique prime factorization results on (tensor products of) group von Neumann algebras $L(G)$ when G is in class $\mathcal{S}$ and has a nonamenable factor as group von Neumann algebra (Theorem I). This result was deduced from unique prime factorization results in [HI17, Theorem B] along with the generalization in [AHHM18, Application 4]. This generalizes results in the discrete setting from [OP04].

One of the main limiting factors for (nontrivial) applications of Theorems H and I is the lack of examples of groups G such that $L(G)$ is a nonamenable factor. As mentioned in the introduction nonamenability criteria for groups acting on trees are provided by [HR19]. An attempt at providing factoriality criteria for such groups was made in [Rau19b], but this paper currently contains a mistake. In general the structure of locally compact group von Neumann algebras is rather poorly understood and many 'natural' groups have amenable or even type I group von Neumann algebras. For instance $L\big(\mathrm{SL}_n(\mathbb{Q}_p)\big)$ is type I [Ber74] and $L(G)$ for G connected is amenable [Con76, Corollary 6.9]. It would therefore be interesting to find more examples of and criteria on locally compact groups under which their group von Neumann algebras would be nonamenable factors. In particular it would be interesting to have weakly amenable locally compact groups G in class $\mathcal{S}$ such that $L(G)$ is a nonamenable factor of type III_λ for every $\lambda \in [0,1]$.

The restrictions above for applying Theorems H and I do not apply to to applications of our Cartan rigidity result Theorem F. Indeed, also for locally compact groups $L^\infty(X) \rtimes G$ is a factor whenever $G \curvearrowright (X,\mu)$ is free and ergodic. As mentioned in the introduction, there are by now many more examples of Cartan rigidity results for discrete groups. It is therefore natural to ask whether these Cartan rigidity results also generalize to the locally compact setting. Also, providing examples of W*-superrigid actions in the locally compact setting would be a very interesting problem. In order for this to succeed, one would also have to prove more OE-rigidity results for locally compact groups.

Appendix A

Proofs of results on cross section equivalence relations

In this appendix, we prove the two 'folklore' Propositions 2.5.38 and 2.5.41. We provide a complete proof to make this thesis as self-contained as possible. We do however not claim originality. The proofs we provide here are similar to the proofs provided in [KPV15, Appendix B], but we have to take of the non-unimodularity of the group G and the fact that the action is not measure preserving.

Recall from Section 2.5 that given a essentially free, nonsingular action $G \curvearrowright (X, \mu)$, the Radon-Nikodym cocycle $D : G \cdot X \to \mathbb{R}_0^+$ is given by

$$D(g, x) = \frac{\mathrm{d}g^{-1} \cdot \mu}{\mathrm{d}\mu}(x)$$

for all $g \in G$ and a.e. $x \in X$. Moreover, we can assume that D is a Borel cocycle in the sense that $D(gh, x) = D(g, hx)D(h, x)$ for all $g, h \in G$ and a.e. $x \in X$. Defining $\omega_r : \mathcal{R} \to \mathbb{C}$ by $\omega_r(x, y) = D(\omega(x, y), y)$, where $\omega : \mathcal{R} \to G$ is the cocycle satisfying $\omega(x, y)y = x$, we get a cocycle $\omega : \mathcal{R} \to \mathbb{R}_0^+$.

Recall Proposition 2.5.38.

Proposition A.1. *Let $G \curvearrowright (X, \mu)$ be a essentially free, non-singular action. Let $X_1 \subseteq X$ be a partial cross section and denote by $\mathcal{R}$ its cross section equivalence relation.*

(a) *There exists a unique measure μ_1 (up to scaling) on X_1 satisfying*

$$(\lambda_G \otimes \mu_1)(\mathcal{W}) = \int_X \sum_{\substack{(g,y)\in\mathcal{W} \\ x=gy}} D(g^{-1}, x)\,\mathrm{d}\mu(x)$$

for all measurable $\mathcal{W} \subseteq G \times X_1$. Moreover, $\mathcal{R}$ is non-singular for μ_1.

(b) *$\mathcal{R}$ is non-singular for μ_1 and its Radon-Nikodym cocycle is given by*

$$D_1(x_1, x_2) = \delta_G\big(\omega(x_1, x_2)\big)\omega_r(x_1, x_2)$$

In particular, if G is unimodular and $G \curvearrowright (X, \mu)$ is measure preserving, then μ_1 is an invariant measure for $\mathcal{R}$.

(c) *If X_1 is a cross section, then $\mathcal{R}$ is ergodic if and only if $G \curvearrowright (X, \mu)$ is ergodic.*

Proof. After discarding a G-invariant null set, we can assume that $G \curvearrowright X$ is free. Define

$$\Psi : G \times X_1 \to X \times X_1 : (g, x) \mapsto (gx, x).$$

Note that Ψ is injective and hence that its image

$$Z = \{(x, y) \in X \times X_1 \mid \exists g \in G : x = gy\} \tag{A.0.1}$$

is Borel. Note moreover that the projection $\pi_\ell : Z \to X : (x, y) \mapsto x$ is countable-to-one since $\pi_\ell \circ \Psi = \theta$. The map Ψ is G-equivariant if we equip Z with the action $G \curvearrowright Z$ given by $g \cdot (x, y) = (gx, y)$ and $G \times X_1$ with the action of left translation on the first component.

Now, define the measure η on Z by

$$\eta(\mathcal{W}) = \int_X \sum_{\substack{y\in X_1 \\ (x,y)\in\mathcal{W}}} \omega_r(y, x)\,\mathrm{d}\mu(x).$$

Note that η is G-invariant and hence so is $\eta \circ \Psi$. Since the Haar measure is unique, it follows that $\eta \circ \Psi = \lambda_G \otimes \mu_1$ for a unique σ-finite measure μ_1. Clearly, μ_1 satisfies the required equation.

Note that

$$\int_G \int_{X_1} f(g, y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}g = \int_X \sum_{(g,y)\in\theta^{-1}(x)} D(g^{-1}, x) f(g, y)\,\mathrm{d}\mu(x),$$

where $\theta : G \times X_1 \to X$ is the action map.

Now, take another partial cross section $X_2 \subseteq X$ and denote by $\mathcal{R}_i$ the cross section equivalence relations for X_i and by μ_i the associated measure. Let Z_i be the set as in (A.0.1) for X_i. Define

$$\mathcal{S} = \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 \in G \cdot x_2\}.$$

Note that $\mathcal{S} = \mathcal{R}(G \curvearrowright X) \cap (X_1 \times X_2)$ is Borel and that the projections $\pi_i : \mathcal{S} \to X_i$ for $i = 1, 2$ are countable-to-one. Define the measures γ_ℓ and γ_r on $\mathcal{S}$ by

$$\nu_\ell(\mathcal{W}) = \int_{X_1} |{}_{x_1}\mathcal{W}| \, \mathrm{d}\mu_1(x_1)$$

and

$$\nu_r(\mathcal{W}) = \int_{X_2} |\mathcal{W}_{x_2}| \, \mathrm{d}\mu_2(x_2)$$

for measurable $\mathcal{W} \subseteq \mathcal{S}$, where as before ${}_{x_1}\mathcal{W} = \{x_2 \in X_2 \mid (x_1, x_2) \in \mathcal{W}\}$ and $\mathcal{W}_{x_2} = \{x_1 \in X \mid (x_1, x_2) \in \mathcal{W}\}$. We claim that $\gamma_\ell = \gamma_r$. Indeed, define

$$\mathcal{Z} = \{(x, x_1, x_2) \in X \times X_1 \times X_2 \mid G \cdot x = G \cdot x_1 = G \cdot x_2\},$$

$$\Phi_1 : G \times \mathcal{S} \to \mathcal{Z} : (g, x_1, x_2) \mapsto (gx_1, x_1, x_2),$$

$$\Phi_2 : G \times \mathcal{S} \to \mathcal{Z} : (g, x_1, x_2) \mapsto (gx_2, x_1, x_2).$$

For $i = 1, 2$, we define the measures ρ_i on $\mathcal{Z}$ by

$$\rho_i(\mathcal{W}) = \int_X \sum_{(x_1, x_2) \in {}_x\mathcal{W}} \omega_r(x_i, x) \, \mathrm{d}\mu(x)$$

for $\mathcal{W} \subseteq \mathcal{Z}$. Note that ρ_1 and ρ_2 belong to the same measure class with Radon-Nikodym derivative

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}\rho_2}(x, x_1, x_2) = \omega_r(x_1, x_2)$$

for a.e. $(x, x_1, x_2) \in \mathcal{Z}$. We have

$$\begin{aligned}
(\lambda_G \otimes \nu_\ell)(\mathcal{W}) &= \int_G \int_{X_1} |{}_{(g, x_1)}\mathcal{W}| \, \mathrm{d}\mu_1(x_1) \, \mathrm{d}g \\
&= \int_X \sum_{(g, x_1) \in \theta^{-1}(x)} D(g^{-1}, x) \, |{}_{(g, x_1)}\mathcal{W}| \, \mathrm{d}\mu(x) \\
&= \int_X \sum_{\substack{(g, x_1, x_2) \in \mathcal{W} \\ gx_1 = x}} \omega_r(x_1, x) \, \mathrm{d}\mu(x)
\end{aligned}$$

$$= \rho_1 \circ \Phi_1(\mathcal{W})$$

for measurable $\mathcal{W} \subseteq G \times \mathcal{S}$ and hence $\lambda_G \otimes \nu_\ell = \rho_1 \circ \Phi_1$. Similarly, $\lambda_G \otimes \nu_r = \rho_2 \circ \Phi_2$. Considering the non-singular Borel automorphism $\zeta : G \times \mathcal{S} \to G \times \mathcal{S}$ given by $\zeta(g, x_1, x_2) = (g\omega(x_1, x_2), x_1, x_2)$ for $g \in G$ and $(x_1, x_2) \in \mathcal{S}$, we have $\Phi_1 = \Phi_2 \circ \zeta$ and hence $\lambda_G \otimes \nu_\ell = \rho_1 \circ \Phi_2 \circ \zeta$ and $\lambda_G \otimes \nu_r = \rho_2 \circ \Phi_2$ belong to the same measure class. It follows that ν_ℓ and ν_r do too. The Radon-Nikodym derivative is given by

$$\begin{aligned} \frac{\mathrm{d}\nu_\ell}{\mathrm{d}\nu_r}(x_1, x_2) &= \frac{\mathrm{d}(\lambda_G \otimes \nu_\ell)}{\mathrm{d}(\lambda_G \otimes \nu_r)}(g, x_1, x_2) \\ &= \frac{\mathrm{d}(\rho_1 \circ \Phi_2 \circ \zeta)}{\mathrm{d}(\rho_1 \circ \Phi_2)}(g, x_1, x_2)\, \frac{\mathrm{d}\rho_1}{\mathrm{d}\rho_2}(gx_2, x_1, x_2) \\ &= \delta_G\big(\omega(x_1, x_2)\big)\omega_r(x_1, x_2). \end{aligned}$$

If $X_2 = X_1$, then $\mathcal{R} = \mathcal{R}_1 = \mathcal{R}_2$. Moreover, ν_ℓ and ν_r are the left and right counting measure on $\mathcal{R}$ respectively. Since they belong to the same measure class, we have that $\mathcal{R}$ is non-singular for the measure μ_1. If G is unimodular and $G \curvearrowright (X, \mu)$ is measure preserving, then $\mathrm{d}\nu_\ell / \mathrm{d}\nu_r = 1$ a.e. and hence μ_1 is an invariant measure.

Now, suppose that X_1 is a cross section, i.e. $G \cdot X_1 = X$ up to null sets. Take Borel maps $\gamma : X \to G$ and $\pi : X \to X_1$ such that $x = \gamma(x)\pi(x)$ for a.e. $x \in X$. A Borel function $f : X \to \mathbb{C}$ is (strictly) G-invariant if and only if it is of the form $f_1 \circ \pi$ for some $\mathcal{R}$-invariant Borel function $f_1 : X_1 \to \mathbb{C}$. From this, it follows immediately that $G \curvearrowright (X, \mu)$ is ergodic if and only if $\mathcal{R}$ is. □

Let X_1 be a partial cross section for $G \curvearrowright (X, \mu)$. We will now prove that $L(\mathcal{R})$ is canonically isomorphic to a corner of the crossed product $M = L^\infty(X) \rtimes G$. Using the same notation as in the proof above, we have that $L^2(Z, \eta)$ is a right $L(\mathcal{R})$ module where

$$(\xi \cdot u_\varphi)(x, y) = \xi\big(x, \varphi(y)\big)\omega_r\big(\varphi(y), y\big)^{1/2} \quad \text{and} \quad (\xi \cdot f)(x, y) = \xi(x, y) f(y)$$

for $\xi \in L^2(Z)$, $\varphi \in [\mathcal{R}]$, $f \in L^\infty(X_1)$ and a.e. $(x, y) \in Z$.

Equip the orbit equivalence relation $\mathcal{R}_G$ with the push-forward of the measure $\lambda_G \otimes \mu$ on $G \times X$ under the Borel isomorphism $\alpha : G \times X \to \mathcal{R}_G : (g, x) \mapsto (gx, x)$. This induces a unitary $U : L^2(\mathcal{R}_G) \to L^2(G \times X)$ given by

$$(U\xi)(g, x) = \xi(gx, x)$$

for $\xi \in L^2(\mathcal{R}_G)$ and a.e. $g \in G$ and a.e. $x \in X$. Denoting by $\tilde{\varphi}$ the dual weight associated to the natural weight on $L^\infty(X)$ given by integrating with respect to μ, the unitary U identifies the M-M-bimodule $L^2(M, \tilde{\varphi})$ with $L^2(\mathcal{R}_G)$ where the left and right module action on the latter is given by

$$(fu_g \cdot \xi)(x,y) = f(x)\xi(g^{-1}x, y)$$

and

$$(\xi \cdot u_g f)(x,y) = \xi(x, gy) f(y) \delta_G(g)^{-1/2}\ D(g,y)^{1/2}.$$

Now, we prove Proposition 2.5.41.

Proposition A.2. *Let $G \curvearrowright (X,\mu)$ be an essentially free, non-singular action. Denote by $M = L^\infty(X) \rtimes G$ and let X_1 be a partial cross section. Then,*

$$pMp \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big),$$

where $\mathcal{U}$ is a neighborhood of identity such that the action map $\mathcal{U} \times X_1 \to X$ is injective and $p = 1_{\mathcal{U}\cdot X_1} \in L^\infty(X)$. In particular, if X_1 is a cross section, then p has central support 1 in M.

Proof. Fix a neighborhood $\mathcal{U}$ of identity such that the action map $\theta : \mathcal{U} \times X_1 \to X$ is injective. Write $X_0 = G \cdot X$. Take $\gamma : X_0 \to G$ and $\pi : X_0 \to X_1$ such that $x = \gamma(x)\pi(x)$ and such that $\gamma(gx) = g$ and $\pi(gx) = x$ for $g \in \mathcal{U}$ and $x \in X_1$. Let $E = \mathcal{U} \cdot X_1$ and denote by $p = 1_E \in L^\infty(X)$ the characteristic function. Using the identification above, we have $L^2(M)p = L^2\big(\mathcal{R}_G \cap (X \times E)\big)$. Consider the map

$$\mathcal{R}_G \cap (X \times E) \to Z \times \mathcal{U} : (x,y) \mapsto \big(x, \pi(y), \gamma(y)\big).$$

This map is a Borel isomorphism with inverse $(x,y,g) \mapsto (x, gy)$. Moreover, using the previous result, this map induces a unitary

$$V : L^2(M)p \to L^2(Z \times \mathcal{U}) : (V\xi)(x,y,g) = \xi(x,gy)\delta_G(g)^{-1/2} D(g,y)^{1/2}.$$

Indeed,

$$\begin{aligned}
\langle V\xi, V\eta\rangle &= \int_Z \int_{\mathcal{U}} (V\xi)(x,y,g)\overline{(V\eta)(x,y,g)}\,\mathrm{d}g\,\mathrm{d}\eta(x,y) \\
&= \int_{\mathcal{U}} \int_Z \xi(x,gy)\overline{\eta(x,gy)}\delta_G(g)^{-1} D(g,y)\,\mathrm{d}\eta(x,y)\,\mathrm{d}g \\
&= \int_{\mathcal{U}} \int_G \int_{X_1} \xi(hy,gy)\overline{\eta(hy,gy)}\delta_G(g)^{-1} D(g,y)\,\mathrm{d}\mu_1(y)\,\mathrm{d}h\,\mathrm{d}g
\end{aligned}$$

$$= \int_G \int_{\mathcal{U}} \int_{X_1} \xi(hgy, gy)\overline{\eta(hgy, gy)} D(g, y) \,\mathrm{d}\mu_1(y) \,\mathrm{d}g \,\mathrm{d}h$$

$$= \int_G \int_X \xi(hx, x)\overline{\eta(hx, x)} \,\mathrm{d}\mu(x) \,\mathrm{d}h$$

$$= \langle \xi, \eta \rangle$$

for $\xi, \eta \in L^2\big(\mathcal{R}_G \cap (X \times E)\big)$, where we used that $\xi(hx, x) = (\xi \circ \alpha)(h, x)$ and similarly for η.

Denote by $\rho : pMp \to B\big(L^2(M)p\big)$ the $*$-antimorphism given by the right action of pMp. Similarly, denote by $\rho : L\mathcal{R} \to B(L^2(Z))$ the $*$-antimorphism given by the right action of $L(\mathcal{R})$. Fix $\varphi_n \in [[\mathcal{R}]]$ such that $\mathcal{R} = \bigcup_n \operatorname{graph} \varphi_n$ and such that the graphs are pairwise disjoint. A direct computation shows that

$$\big(V\rho(pu_g p)V^* \xi\big)(x, y, h) =$$
$$\xi\big(x, \pi(ghy), \gamma(ghy)\big)\delta_G\big(\gamma(ghy)^{-1} gh\big)^{-1/2} D\big(\gamma(ghy)^{-1} gh, y\big)^{1/2} 1_E(ghy)$$

for all $a \in L^\infty(X)$ all $g \in G$, a.e. $(x, y) \in X$ and a.e. $h \in G$. Write

$$A_n = \{(y, g) \in X_1 \times \mathcal{U} \mid ghy \in E \text{ and } \pi(ghy) = \varphi_n(y)\}$$

and denote $u_n = u_{\varphi_n}$. Then,

$$V\rho(pu_g p)V^* = \sum_n (\rho \otimes \mathrm{id})(1_{A_n}) V_n \,\big(\rho(u_n) \otimes \lambda_g^*\big) \in \rho(L\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big),$$

where $V_n \in \rho\big(L^\infty(X_1)\big) \mathbin{\overline{\otimes}} R(G)$ is defined by $V_n(y) = \rho_{\omega(y, \varphi_n(y))}$ for $y \in X_1$. Similarly,

$$\big(V\rho(pap)V^* \xi\big)(x, y, h) = \xi(x, y, h) a(hy)$$

for all $a \in L^\infty(X)$ all $g \in G$ and a.e. $(x, y) \in X$. Hence $V\rho(pap)V^* \in \rho(L\mathcal{R}) \mathbin{\overline{\otimes}} L^\infty(\mathcal{U})$. It follows that

$$V\rho(pMp)V^* \subseteq \rho(L\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big).$$

Denote by $\gamma : M \to B\big(L^2(M)p\big)$ the $*$-morphism given by the left action of M. A straightforward computation shows that $V\gamma(au_g)V^*$ commutes with $\rho(L\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big)$ for all $a \in L^\infty(X)$ and all $g \in G$. Since

$$B\big(L^2(M)p\big) \cap \gamma(M)' = \rho(pMp),$$

we have that

$$V^*\big(\rho(L\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big)\big)V \subseteq \gamma(M)' \cap B\big(L^2(M)p\big) = \rho(pMp)$$

and hence $V\rho(pMp)V^* = \rho(L\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big)$. Since ρ is a faithful anti-representation, we get that

$$pMp \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B\big(L^2(\mathcal{U})\big).$$

□

Bibliography

[Ada94] S. Adams, Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups. *Topology* **33** (1994), 765–783.

[AO75] C. A. Akemann and P. A. Ostrand, On a tensor product C*-algebra associated with the free group on two generators. *J. Math. Soc. Jpn.* **27** (1975), 589–599.

[APT73] C. A. Akemann, G. K. Pedersen, and J. Tomiyama, Multipliers of C*-algebras. *J. Funct. Anal.* **13** (1973), 277–301.

[Ana79] C. Anantharaman-Delaroche, Action moyennable d'un groupe localement compact sur une algebre de von Neumann. *Math. Scand.* **45** (1979), 289–304.

[Ana82] C. Anantharaman-Delaroche, Action moyennable d'un groupe localement compact sur une algèbre de von Neumann II. *Math. Scand.* **50** (1982), 251–268.

[Ana87] C. Anantharaman-Delaroche, Systèmes dynamiques non commutatifs et moyennabilité. *Math. Ann.* **279** (1987), 297–315.

[Ana95] C. Anantharaman-Delaroche, Amenable correspondences and approximation properties for von Neumann algebras. *Pac. J. Math.* **171** (1995), 309–341.

[Ana02] C. Anantharaman-Delaroche, Amenability and exactness for dynamical systems and their C*-algebras. *T. Am. Math. Soc.* **354** (2002), 4153–4179.

[AP14] C. Anantharaman-Delaroche and S. Popa, An introduction to II_1 factors. 2014. URL: http://www.math.ucla.edu/~popa/Books/IIun-v13.pdf.

[AR00] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids. *Monographie de l'Enseignement Mathématique* **36**. L'Enseignement Mathématique, Genève, 2000.

[AHHM18] H. Ando, U. Haagerup, C. Houdayer, and A. Marrakchi, Structure of bicentralizer algebras and inclusions of type III factors. 2018. *Preprint.* arXiv: 1804.05706.

[AD08] G. Arzhantseva and T. Delzant, Examples of random groups. 2008. *Preprint.* Available at: http://www.unige.ch/math/folks/arjantse/Abs/random.pdf.

[BS93] S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres. *Ann. Sci. Ecole Norm. S.* **4** (1993), 425–488.

[BSV03] S. Baaj, G. Skandalis, and S. Vaes, Non-semi-regular quantum groups coming from number theory. *Commun. Math. Phys.* **235** (2003), 139–167.

[BFS13] U. Bader, A. Furman, and R. Sauer, Integrable measure equivalence and rigidity of hyperbolic lattices. *Invent. Math.* **194** (2013), 313–379.

[BdlH86] E. Bédos and P. de la Harpe, Moyennabilité intérieure des groupes: définitions et exemples. *L'enseignement Mathématique* **32** (1986), 139–157.

[BdlHV08] B. Bekka, P. de la Harpe, and A. Valette, Kazhdan's Property (T). *New Mathematical Monographs* **11**. Cambridge University Press, Cambridge, 2008.

[Ber74] J. Bernstein, All reductive p-adic groups of type I. *Funct. Anal. Its Appl.* **8** (1974), 91–93.

[Bla06] B. Blackadar, Operator Algebras. *Encyclopaedia of Mathematical Sciences* **122**. Springer, Berlin, 2006.

[BHR14] R. Boutonnet, C. Houdayer, and S. Raum, Amalgamated free product type III factors with at most one Cartan subalgebra. *Compos. Math.* **150** (2014), 143–174.

[BHV18] R. Boutonnet, C. Houdayer, and S. Vaes, Strong solidity of free Araki-Woods factors. *Am. J. Math.* **140** (2018), 1231–1252.

[BF84] M. Bozejko and G. Fendler, Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. *B. Unione Mat. Ital.* **3** (1984), 297–302.

[BH99] M. R. Bridson and A. Haefliger, Metric Spaces of Non-Positive Curvature. *Grundlehren der mathematischen Wissenschaften* **319**. Springer, Berlin, 1999.

[BCL17] J. Brodzki, C. Cave, and K. Li, Exactness of locally compact groups. *Adv. Math.* **312** (2017), 209–233.

[BDV18] A. Brothier, T. Deprez, and S. Vaes, Rigidity for von Neumann algebras given by locally compact groups and their crossed products. *Commun. Math. Phys.* **361** (2018), 85–125.

[BO08] N. Brown and N. Ozawa, C^*-Algebras and Finite-Dimensional Approximations. *Graduate Studies in Mathematics* **88**. American Mathematical Society, Providence, 2008.

[CCMT15] P.-E. Caprace, Y. Cornulier, N. Monod, and R. Tessera, Amenable hyperbolic groups. *J. Eur. Math. Soc.* **17** (2015), 2903–2947.

[CP13] I. Chifan and J. Peterson, Some unique group-measure space decomposition results. *Duke Math. J.* **162** (2013), 1923–1966.

[CS13] I. Chifan and T. Sinclair, On the structural theory of II_1 factors of negatively curved groups. *Ann. Sci. Ecole Norm. S.* **46** (2013), 1–33.

[CSU13] I. Chifan, T. Sinclair, and B. Udrea, On the structural theory of II_1 factors of negatively curved groups, II: Actions by product groups. *Adv. Math.* **245** (2013), 208–236.

[Cho83] M. Choda, Group factors of the Haagerup type. *Proc. Jpn. Acad. A-Math.* **59** (1983), 174–177.

[CE76] M.-D. Choi and E. G. Effros, The Completely Positive Lifting Problem for C^*-algebras. *Ann. Math.* **104** (1976), 585.

[Con73] A. Connes, Une classification des facteurs de type III. *Ann. Sci. Ecole Norm. S.* **6** (1973), 133–252.

[Con75a] A. Connes, Outer conjugacy classes of automorphisms of factors. *Ann. Sci. Ecole Norm. S.* **8** (1975), 383–419.

[Con75b] A. Connes, On hyperfinite factors of type III_0 and Krieger's factors. *J. Funct. Anal.* **18** (1975), 318–327.

[Con76] A. Connes, Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. *Ann. Math.* **104** (1976), 73–115.

[Con82] A. Connes, Classification des facteurs. In *Oper. Algebr. Appl. Part 2*. Ed. by R. V. Kadison. Vol. 38. *Proceedings of Symposia in Pure Mathematics*. Providence, 1982, 43–109.

[Con94] A. Connes, Noncommutative Geometry. Academic Press, San Diego, 1994.

[CJ85] A. Connes and V. Jones, Property T for von Neumann algebras. *B. Lon. Math. Soc.* **17** (1985), 57–62.

[CS78] A. Connes and E. Størmer, Homogeneity of the state space of factors of type III_1. *J. Funct. Anal.* **28** (1978), 187–196.

[CT77] A. Connes and M. Takesaki, The flow of weights on factors of type III. *Tohoku Math. J.* **29** (1977), 473–575.

[Con85] J. B. Conway, A Course in Functional Analysis. *Graduate Texts in Mathematics* **96**. Springer, New York, 1985.

[Cor17] Y. Cornulier, Locally compact wreath products. To appear in *J. Aust. Math. Soc.*

[CH89] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.* **96** (1989), 507–549.

[Dav96] K. R. Davidson, C^*-algebras by example. American Mathematical Society, Providence, 1996.

[Day57] M. M. Day, Amenable semigroups. *Illinois J. Math.* **1** (1957), 509–544.

[dCES79] J. de Cannière, M. Enock, and J.-M. Schwartz, Algèbres de Fourier associées à une algèbre de Kac. *Math. Ann.* **245** (1979), 1–22.

[dCH85] J. de Cannière and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. *Am. J. Math.* **107** (1985), 455–500.

[DE14] A. Deitmar and S. Echterhoff, Principles of Harmonic Analysis. Second Edi. *Universitext.* Springer, Berlin, 2014.

[DL14] S. Deprez and K. Li, Permanence properties of property A and coarse embeddability for locally compact groups. 2014. *Preprint.* arXiv: 1403.7111.

[DL15] S. Deprez and K. Li, Property A and uniform embedding for locally compact groups. *J. Noncommut. Geom.* **9** (2015), 797–819.

[Dep19] T. Deprez, Ozawa's class $\mathcal{S}$ for locally compact groups and unique prime factorization of group von Neumann algebras. To appear in *P. Roy. Soc. Edinb. A.*

[DV18] T. Deprez and S. Vaes, Inner amenability, property Gamma, McDuff II_1 factors and stable equivalence relations. *Ergod. Theor. Dyn. Syst.* **38** (2018), 2618–2624.

[Die60] J. Dieudonné, Sur le produit de composition II. *J. Math. Pure. Appl.* **39** (1960), 275–292.

[Dix53] J. Dixmier, Formes linéaires sur un anneau d'opérateurs. *B. Soc. Math. Fr.* **79** (1953), 9–39.

[Dor93] B. Dorofaeff, The Fourier algebra of $\mathrm{SL}(2,\mathbb{R})$, $n \geqslant 2$, has no multiplier bounded approximate unit. *Math. Ann.* **297** (1993), 707–724.

[Dor96] B. Dorofaeff, Weak amenability and semidirect products in simple Lie groups. *Math. Ann.* **306** (1996), 737–742.

[DS58] N. Dunford and J. Schwartz, Linear Operatos, Part 1: General Theory. Interscience Publishers, New York, 1958.

[Dye59] H. A. Dye, On groups of measure preserving transformations. I. *Am. J. Math.* **81** (1959), 119–159.

[Dye63] H. A. Dye, On groups of measure preserving transformations. II. *Am. J. Math.* **85** (1963), 551.

[Eff75] E. G. Effros, Property Γ and inner amenability. *Proc. Am. Math. Soc.* **47** (1975), 483–486.

[Eym64] P. Eymard, L'algèbre de Fourier d'un groupe localement compact. *B. Soc. Math. Fr.* **92** (1964), 181–236.

[FM77a] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. I. *T. Am. Math. Soc.* **234** (1977), 289–324.

[FM77b] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. II. *T. Am. Math. Soc.* **234** (1977), 325–359.

[Føl55] E. Følner, On groups with full Banach mean value. *Math. Scand.* **3** (1955), 243–254.

[For74] P. Forrest, On the virtual groups defined by ergodic actions of $\mathbb{R}^n$ and $\mathbb{Z}^n$. *Adv. Math.* **14** (1974), 271–308.

[Gab00] D. Gaboriau, Coût des relations d'équivalence et des groupes. *Invent. Math.* **139** (2000), 41–98.

[GIT16] D. Gaboriau, A. Ioana, and R. Tucker-Drob, Cocycle superrigidity for translation actions of product groups. 2016. *Preprint.* arXiv: 1603.07616.

[Ge98] L. Ge, Applications of free entropy to finite von Neumann algebras, II. *Ann. Math.* **147** (1998), 143–157.

[God51] R. Godement, Mémoire sur la théorie des caractères dans les groupes localement compacts unimodulaires. *J. Math. Pure. Appl.* **30** (1951), 1–110.

[Gre69] F. Greenleaf, Invariant means on topological groups and their applications. *Van Nostrand Mathematical Studies Series* **16**. Van Nostrand Reinhold Company, New York, 1969.

[Gri85] R. I. Grigorchuk, Degrees of growth of finitely generated groups, and the theory of invariant means. *Math. USSR Izv+* **25** (1985), 259–300.

[Gro87] M. Gromov, Hyperbolic Groups. In *Essays Gr. Theory*. Springer, New York, 1987, 75–263.

[Gro93] M. Gromov, Geometric Group Theory, volume 2: Asymptotic Invariants of Infinite roups. *London Mathematical Society Lecture Note Series* **182**. Cambridge University Press, Cambridge, 1993.

[Gro03] M. Gromov, Random walk in random groups. *Geom. Funct. Anal.* **13** (2003), 73–146.

[GHW05] E. Guentner, N. Higson, and S. Weinberger, The Novikov conjecture for linear groups. *Publ. Math.* **101** (2005), 243–268.

[Haa75] U. Haagerup, The standard form of von Neumann algebras. *Math. Scand.* **37** (1975), 271–283.

[Haa78] U. Haagerup, An example of a non nuclear C^*-algebra, which has the metric approximation property. *Invent. Math.* **50** (1978), 279–293.

[Haa87] U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type III_1. *Acta Math.* **158** (1987), 95–148.

[Haa88] U. Haagerup, Tomita-Takesaki Theory for Pedestrians. 1988.

[Haa16] U. Haagerup, Group C^*-algebras without the completely bounded approximation property. 2016. *Preprint.* arXiv: 1603.00209.

[HK94] U. Haagerup and J. Kraus, Approximation properties for group C^*-algebras and group von Neumann algebras. *T. Am. Math. Soc.* **344** (1994), 667–699.

[HT67] J. Hakeda and J. Tomiyama, On some extension properties of von Neumann algebras. *Tohoku Math. J.* **19** (1967), 315–323.

[Han90] M. L. Hansen, Weak amenability of the universal covering group of $SU(1, n)$. *Math. Ann.* **288** (1990), 445–472.

[HI17] C. Houdayer and Y. Isono, Unique prime factorization and bicentralizer problem for a class of type III factors. *Adv. Math.* **305** (2017), 402–455.

[HPV13] C. Houdayer, S. Popa, and S. Vaes, A class of groups for which every action is W^*-superrigid. *Group. Geom. Dynam.* **7** (2013), 577–590.

[HR19] C. Houdayer and S. Raum, Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras. *Comment. Math. Helv.* **94** (2019), 185–219.

[HU15] C. Houdayer and Y. Ueda, Asymptotic structure of free product von Neumann algebras. *Math. Proc. Cambridge* **161** (2015), 489–516.

[HV13] C. Houdayer and S. Vaes, Type III factors with unique Cartan decomposition. *J. Math. Pure. Appl.* **100** (2013), 564–590.

[Hur83] T. Huruya, On compact completely bounded maps of C*-algebras. *Mich. Math. J.* **30** (1983), 213–220.

[Ioa11] A. Ioana, W*-superrigidity for Bernoulli actions of property (T) groups. *J. Am. Math. Soc.* **24** (2011), 1175–1226.

[Ioa13] A. Ioana, Classification and rigidity for von Neumann algebras. In *Eur. Congr. Math. Krakow*. European Mathematical Society Publishing House, Zürich, 2013, 601–625.

[Ioa15] A. Ioana, Cartan subalgebras of amalgamated free product II_1 factors. *Ann. Sci. Ecole Norm. S.* **48** (2015), 71–130.

[Ioa17] A. Ioana, Rigidity for von Neumann algebras. 2017. *Preprint*. arXiv: 1712.00151.

[Iso19] Y. Isono, Unitary conjugacy for type III subfactors and W*-superrigidity. 2019. *Preprint*. arXiv: 1902.01049.

[Jol15] P. Jolissaint, Proper cocycles and weak forms of amenability. *Colloq. Math.* **138** (2015), 73–87.

[Jon83] V. Jones, Index for subfactors. *Invent. Math.* **72** (1983), 1–25.

[Kai04] V. A. Kaimanovich, Boundary amenability of hyperbolic spaces. In *Discret. Geom. Anal.* Vol. 347. *Contemporary Mathematics*. American Mathematical Society, Providence, 2004, 83–111.

[Kak38] S. Kakutani, Two fixed-point theorems concerning bicompact convex sets. *Proc. Imp. Acad.* **14** (1938), 242–245.

[Kec95] A. S. Kechris, Classical Descriptive Set Theory. *Graduate Texts in Mathematics* **156**. Springer, New York, 1995.

[KW99a] E. Kirchberg and S. Wassermann, Permanence properties of C*-exact groups. *Doc. Math.* **4** (1999), 513–558.

[KW99b] E. Kirchberg and S. Wassermann, Exact groups and continuous bundles of C*-algebras. *Math. Ann.* **315** (1999), 169.

[KKR17] J. Koivisto, D. Kyed, and S. Raum, Measure equivalence and coarse equivalence for unimodular locally compact groups. 2017. *Preprint*. arXiv: 1703.08121.

[KKR18] J. Koivisto, D. Kyed, and S. Raum, Measure equivalence for non-unimodular groups. 2018. *Preprint*. arXiv: 1805.09063.

[Kri69a] W. Krieger, On non-singular transformations of a measure space. I. *Z. Wahrscheinlichkeit* **11** (1969), 83–97.

[Kri69b] W. Krieger, On non-singular transformations of a measure space. II. *Z. Wahrscheinlichkeit* **11** (1969), 98–119.

[KV03] J. Kustermans and S. Vaes, Locally compact quantum groups in the von Neumann algebraic setting. *Math. Scand.* **92** (2003), 68–92.

[KPV15] D. Kyed, H. D. Petersen, and S. Vaes, L^2-Betti numbers of locally compact groups and their cross section equivalence relations. *T. Am. Math. Soc.* **367** (2015), 4917–4956.

[Lan95] E. C. Lance, Hilbert C*-Modules. *London Mathematical Society Lecture Note Series* **210**. Cambridge University Press, Cambridge, 1995.

[LP91] A. T.-M. Lau and A. L. T. Paterson, Inner amenable locally compact groups. *T. Am. Math. Soc.* **325** (1991), 155–169.

[LR87] V. Losert and H. Rindler, Conjugation-invariant means. *Colloq. Math.* **51** (1987), 221–225.

[Mac52] G. W. Mackey, Induced representations of locally compact groups I. *Ann. Math.* **55** (1952), 101–139.

[Mac62] G. W. Mackey, Point realizations of transformation groups. *Illinois J. Math.* **6** (1962), 327–335.

[Mah64] D. Maharam, Incompressible transformations. *Fund. Math.* **56** (1964), 35–50.

[Mar36] A. Markov, Quelques théorèmes sur les ensembles abéliens. *Dokl. Akad. Nauk SSSR* **1** (1936), 311–313.

[MRV13] N. Meesschaert, S. Raum, and S. Vaes, Stable orbit equivalence of Bernoulli actions of free groups and isomorphism of some of their factor actions. *Expo. Math.* **31** (2013), 274–294.

[MMS04] I. Mineyev, N. Monod, and Y. Shalom, Ideal bicombings for hyperbolic groups and applications. *Topology* **43** (2004), 1319–1344.

[MS04] N. Monod and Y. Shalom, Cocycle superrigidity and bounded cohomology for negatively curved spaces. *J. Differ. Geom.* **67** (2004), 395–455.

[Mun00] J. R. Munkres, Topology. 2nd editio. Pearson, 2000, 537.

[MvN37] F. J. Murray and J. von Neumann, On rings of operators. II. *T. Am. Math. Soc.* **41** (1937), 208–248.

[MvN36] F. J. Murray and J. von Neumann, On rings of operators. *Ann. Math.* **37** (1936), 116–229.

[MvN43] F. J. Murray and J. von Neumann, On rings of operators. IV. *Ann. Math.* **44** (1943), 716–808.

[Nam66] I. Namioka, On a recent theorem by H. Reiter. *Proc. Am. Math. Soc.* **17** (1966), 1101–1102.

[Ols80] A. Y. Ol'shanskii, On the problem of the existence of an invariant mean on a group. *Russ. Math. Surv.* **35** (1980), 180–181.

[OV94] A. Onishchik and E. Vinberg, Lie Groups and Lie Algebras III: Structyure of Lie Groups and Lie Algebras. Ed. by A. L. Onishchik and E. B. Vinberg. *Encyclopaedia of Mathematical Sciences* **41**. Springer, Berlin, 1994.

[Osa14] D. Osajda, Small cancellation labellings of some infinite graphs and applications. 2014. *Preprint.* arXiv: 1406.5015.

[Oza00] N. Ozawa, Amenable actions and exactness for discrete groups. *C. R. Acad. Sci. I-Math.* **330** (2000), 691–695.

[Oza04] N. Ozawa, Solid von Neumann algebras. *Acta Math.* **192** (2004), 111–117.

[Oza06] N. Ozawa, A Kurosh-type theorem for type II_1 factors. *Int. Math. Res. Notices* (2006), 1–21.

[Oza07] N. Ozawa, Weak amenability of hyperbolic groups. 2007. *Preprint.* arXiv: 0704.1635.

[Oza09] N. Ozawa, An example of a solid von Neumann algebra. *Hokkaido Math. J.* **38** (2009), 557–561.

[Oza12] N. Ozawa, Examples of groups which are not weakly amenable. *Kyoto J. Math.* **52** (2012), 333–344.

[OP04] N. Ozawa and S. Popa, Some prime factorization results for type II_1 factors. *Invent. Math.* **156** (2004), 223–234.

[OP10a] N. Ozawa and S. Popa, On a class of II_1 factors with at most one Cartan subalgebra, II. *Am. J. Math.* **132** (2010), 841–866.

[OP10b] N. Ozawa and S. Popa, On a class of II_1 factors with at most one Cartan subalgebra. *Ann. Math.* **172** (2010), 713–749.

[Pat88] A. L. Paterson, Amenability. *Mathematical Surverys and Monographs* **29**. American Mathematical Society, Providence, 1988.

[Pau84] V. I. Paulsen, Every completely polynomially bounded operator is similar to a contraction. *J. Funct. Anal.* **55** (1984), 1–17.

[Pau03] V. I. Paulsen, Completely Bounded Maps and Operator Algebras. *Cambridge Studies in Advanced Mathematics* **78**. Cambridge University Press, Cambridge, 2003.

[Pet10] J. Peterson, Examples of group actions which are virtually W*-superrigid. 2010. *Preprint.* arXiv: 1002.1745.

[Pop83] S. Popa, Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras. *J. Operat. Theor.* **9** (1983), 253–268.

[Pop86] S. Popa, Correspondences. 1986.

[Pop06a] S. Popa, On a class of type II_1 factors with Betti numbers invariants. *Ann. Math.* **163** (2006), 809–899.

[Pop06b] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, I. *Invent. Math.* **165** (2006), 369–408.

[Pop06c] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, II. *Invent. Math.* **165** (2006), 409–451.

[Pop07] S. Popa, Deformation and rigidity for group actions and von Neumann algebras. In *Proc. Int. Congr. Math. Madrid, August 22-30, 2006.* European Mathematical Society Publishing House, Zürich, 2007, 445–477.

[PV10] S. Popa and S. Vaes, Group measure space decomposition of II_1 factors and W*-superrigidity. *Invent. Math.* **182** (2010), 371–417.

[PV11] S. Popa and S. Vaes, Cocycle and orbit superrigidity for lattices in $\mathrm{SL}(n, \mathbb{R})$ acting on homogeneous spaces. In *Geom. rigidity, Gr. actions.* The University of Chicago Press, Chicago, 2011, 419–451.

[PV14a] S. Popa and S. Vaes, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups. *Acta Math.* **212** (2014), 141–198.

[PV14b] S. Popa and S. Vaes, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups. *J. Reine Angew. Math.* **694** (2014), 141–198.

[Rau19a] S. Raum, C*-simplicity after Breuillard, Haagerup, Kalantar, Kennedy and Ozawa. 2019. URL: http://raum-brothers.eu/sven/data/publications/bourbaki.pdf.

[Rau19b] S. Raum, C*-simplicity of locally compact Powers groups. *J. Reine Angew. Math.* **748** (2019), 173–205.

[Rei64] H. Reiter, Sur la propriété (P_1) et les fonctions de type positif. *C. R. Acac. Sci. Paris* **258** (1964), 5134–51355.

[Rei65] H. Reiter, On some properties of locally compact groups. *Indagat. Math.* **68** (1965), 697–701.

[Run02] V. Runde, Lectures on Amenability. *Lecture Notes in Mathematics* **1774**. Springer, Berlin, 2002.

[Sak56] S. Sakai, A characterization of W*-algebras. *Pac. J. Math.* **6** (1956), 763–773.

[Sak09a] H. Sako, The class $\mathcal{S}$ as an ME invariant. *Int. Math. Res. Notices* (2009), 2749–2759.

[Sak09b] H. Sako, Measure equivalence rigidity and bi-exactness of groups. *J. Funct. Anal.* **257** (2009), 3167–3202.

[Sau77] J.-L. Sauvageot, Sur le type du produit croisé d'une algèbre de von Neumann par un groupe localement compact. *B. Soc. Math. Fr.* **105** (1977), 349–368.

[Sau85] J.-L. Sauvageot, Implementation canonique pour les co-actions d'un groupe localement compact et les systèmes dynamiques généralisés. *Math. Ann.* **270** (1985), 325–337.

[Sch63] J. Schwartz, Two finite, non-hyperfinite, non-isomorphic factors. *Commun. Pur. Appl. Math.* **16** (1963), 19–26.

[Ser80] J.-P. Serre, Trees. Springer, Berlin, Heidelberg, 1980.

[She50] S. Sherman, The second adjoint of a C*-algebra. In *Proc. Int. Congr. Math.* **1**. Cambridge, 1950, 470.

[Sin55] I. M. Singer, Automorphisms of finite factors. *Am. J. Math.* **77** (1955), 117–133.

[Ska88] G. Skandalis, Une notion de nucléarité en K-théorie (d'après J. Cuntz). *K-Theory* **1** (1988), 549–573.

[Sti55] W. F. Stinespring, Positive functions on C*-algebras. *Proc. Am. Math. Soc.* **6** (1955), 211–216.

[Str74] R. A. Struble, Metrics in locally compact groups. *Compos. Math.* **28** (1974), 217–222.

[Sut78] C. Sutherland, Type analysis of the regular representation of a nonunimodular group. *Pac. J. Math.* **79** (1978), 225–250.

[Suz16] Y. Suzuki, Elementary constructions of non-discrete C*-simple groups. *Proc. Am. Math. Soc.* **145** (2016), 1369–1371.

[Szw91] R. Szwarc, Groups acting on trees and approximation properties of the Fourier algebra. *J. Funct. Anal.* **95** (1991), 320–343.

[Tak54] Z. Takeda, Conjugate spaces of operator algebras. *Proc. Jpn. Acad.* **30** (1954), 90–95.

[Tak70] M. Takesaki, Tomita's Theory of Modular Hilbert Algebras and its Applications. *Lecture Notes in Mathematics* **128**. Springer, Berlin, 1970.

[Tak73] M. Takesaki, Duality for crossed products and the structure of von Neumann algebras of type III. *Acta Math.* **131** (1973), 249–310.

[Tak79] M. Takesaki, Theory of Operator Algebras I. **124**. Springer, New York, 1979.

[Tak03a] M. Takesaki, Theory of Operator Algebras II. *Encyclopaedia of Mathematical Sciences* **125**. Springer, Berlin, 2003.

[Tak03b] M. Takesaki, Theory of Operator Algebras III. *Encyclopaedia of Mathematical Sciences* **127**. Springer, Berlin, 2003.

[Tom67] M. Tomita, On canonical forms of von Neumann algebras. In *Fifth Funct. Anal. Symp.* Tôhuko University, Sendai, 1967, 101–102.

[Ued13] Y. Ueda, Some analysis of amalgamated free products of von Neumann algebras in the non-tracial setting. *J. London Math. Soc.* **88** (2013), 25–48.

[Vae01] S. Vaes, The unitary implementation of a locally compact quantum group action. *J. Funct. Anal.* **180** (2001), 426–480.

[Vae11] S. Vaes, Rigidity for von Neumann algebras and their invariants. In *Proc. Int. Congr. Math. 2010 (ICM 2010)*, **III**. Hindustan Book Agency, 2011, 1624–1650.

[Vae13] S. Vaes, One-cohomology and the uniqueness of the group measure space decomposition of a II_1 factor. *Math. Ann.* **355** (2013), 661–696.

[Vae16] S. Vaes, Amenability versus nonamenability: an introduction to von Neumann algebras. In *Eur. Congr. Math.* European Mathematical Society, Berlin, 2016.

[VV07] S. Vaes and R. Vergnioux, The boundary of universal discrete quantum groups, exactness, and factoriality. *Duke Math. J.* **140** (2007), 35–84.

[Van05] A. Van Daele, Locally compact quantum groups: the von Neumann algebra versus the C^*-algebra approach. *B. Kerala Math. Assoc.* **2005** (2005), 153–177.

[Voi96] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory III: the absence of Cartan subalgebras. *Geom. Funct. Anal.* **6** (1996), 172–199.

[vNeu29] J. von Neumann, Zur allgemeinen theorie des masses. *Fund. Math.* **13** (1929), 73–116.

[vNeu40] J. von Neumann, On rings of operators. III. *Ann. Math.* **41** (1940), 94–161.

[vNeu49] J. von Neumann, On rings of operators. Reduction theory. *Ann. Math.* **50** (1949), 401–485.

[Wil07] D. P. Williams, Crossed Products of C^*-Algebras. *Mathematical Surveys and Monographs* **134**. American Mathematical Society, Providence, 2007.

[Zim76] R. J. Zimmer, Extensions of ergodic group actions. *Illinois J. Math.* **20** (1976), 373–409.

[Zim78] R. J. Zimmer, Amenable ergodic group actions and an application to Poisson boundaries of random walks. *J. Funct. Anal.* **27** (1978), 350–372.

[Zim84] R. J. Zimmer, Ergodic Theory and Semisimple Groups. Birkhäuser, Boston, 1984, 209.

List of publications

- T. Deprez and S. Vaes, Inner amenability, property Gamma, McDuff II_1 factors and stable equivalence relations. *Ergod. Theor. Dyn. Syst.* **38** (2018), 2618–2624

- A. Brothier, T. Deprez, and S. Vaes, Rigidity for von Neumann algebras given by locally compact groups and their crossed products. *Commun. Math. Phys.* **361** (2018), 85–125

- T. Deprez, Ozawa's class $\mathcal{S}$ for locally compact groups and unique prime factorization of group von Neumann algebras. To appear in *P. Roy. Soc. Edinb. A*

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FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
SECTION OF ANALYSIS
Celestijnenlaan 200B box 2400
B-3001 Leuven
tobe.deprez@kuleuven.be